CHAPTER 13

Complex Numbers and Functions. Complex Differentiation

The transition from "real calculus" to "complex calculus" starts with a discussion of ***complex numbers*** and their geometric representation in the ***complex plane***. We then progress to ***analytic functions*** in Sec. 13.3. We desire functions to be analytic because these are the "useful functions" in the sense that they are differentiable in some domain and operations of complex analysis can be applied to them. The most important equations are therefore the Cauchy–Riemann equations in Sec. 13.4 because they allow a test of analyticity of such functions. Moreover, we show how the Cauchy–Riemann equations are related to the important ***Laplace equation***.

The remaining sections of the chapter are devoted to elementary complex functions (exponential, trigonometric, hyperbolic, and logarithmic functions). These generalize the familiar real functions of calculus. Detailed knowledge of them is an absolute necessity in practical work, just as that of their real counterparts is in calculus.

Prerequisite: Elementary calculus.
References and Answers to Problems: App. 1 Part D, App. 2.

13.1 Complex Numbers and Their Geometric Representation

The material in this section will most likely be familiar to the student and serve as a review.

Equations without *real* solutions, such as $x^2 = -1$ or $x^2 - 10x + 40 = 0$, were observed early in history and led to the introduction of complex numbers.[1] By definition, a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by CARL FRIEDRICH GAUSS (see the footnote in Sec. 5.4), who also paved the way for a general use of complex numbers.

x is called the **real part** and y the **imaginary part** of z, written

$$x = \text{Re}\, z, \qquad y = \text{Im}\, z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

(0, 1) is called the **imaginary unit** and is denoted by i,

$$i = (0, 1). \tag{1}$$

Addition, Multiplication. Notation $z = x + iy$

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2). \tag{2}$$

Multiplication is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1). \tag{3}$$

These two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for real numbers x_1, x_2. Hence the complex numbers "***extend***" the real numbers. We can thus write

$$(x, 0) = x. \qquad \text{Similarly,} \qquad (0, y) = iy$$

because by (1), and the definition of multiplication, we have

$$iy = (0, 1)y = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, \quad 0 \cdot 0 + 1 \cdot y) = (0, y).$$

Together we have, by addition, $(x, y) = (x, 0) + (0, y) = x + iy$.

In practice, complex numbers $z = (x, y)$ are written

$$z = x + iy \tag{4}$$

or $z = x + yi$, e.g., $17 + 4i$ (instead of $i4$).

Electrical engineers often write j instead of i because they need i for the current.

If $x = 0$, then $z = iy$ and is called **pure imaginary**. Also, (1) and (3) give

$$i^2 = -1 \tag{5}$$

because, by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives the following very simple recipe. Multiply each term by each other term and use $i^2 = -1$ when it occurs [see (3)]:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$

This agrees with (3). And it shows that $x + iy$ is a more practical notation for complex numbers than (x, y).

If you know vectors, you see that (2) is vector addition, whereas the multiplication (3) has no counterpart in the usual vector algebra.

EXAMPLE 1 **Real Part, Imaginary Part, Sum and Product of Complex Numbers**

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$. Then $\text{Re}\, z_1 = 8$, $\text{Im}\, z_1 = 3$, $\text{Re}\, z_2 = 9$, $\text{Im}\, z_2 = -2$ and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i.$$ ■

Subtraction, Division

Subtraction and **division** are defined as the inverse operations of addition and multiplication, respectively. Thus the **difference** $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Hence by (2),

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{6}$$

The **quotient** $z = z_1/z_2$ $(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain $x_1 = x_2x - y_2y$, $y_1 = y_2x + x_2y$. The solution is

$$z = \frac{z_1}{z_2} = x + iy, \qquad x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{7*}$$

The ***practical rule*** used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - iy_2$ and simplifying:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{7}$$

EXAMPLE 2 **Difference and Quotient of Complex Numbers**

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$ we get $z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$ and

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}i.$$

Check the division by multiplication to get $8 + 3i$. ■

Complex numbers satisfy the same commutative, associative, and distributive laws as real numbers (see the problem set).

Complex Plane

So far we discussed the algebraic manipulation of complex numbers. Consider the geometric representation of complex numbers, which is of great practical importance. We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis**, and the vertical y-axis, called the **imaginary axis**. On both axes we choose the same unit of length (Fig. 318). This is called a **Cartesian coordinate system**.

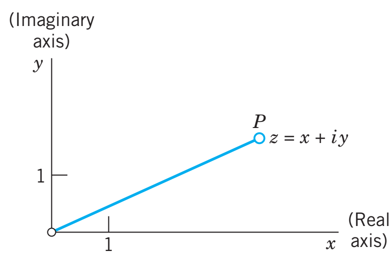


Fig. 318. The complex plane

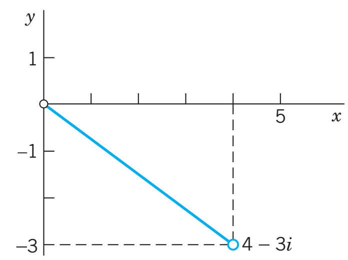


Fig. 319. The number $4 - 3i$ in the complex plane

We now plot a given complex number $z = (x, y) = x + iy$ as the point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane**.[2] Figure 319 shows an example.

Instead of saying "the point represented by z in the complex plane" we say briefly and simply "***the point z in the complex plane***." This will cause no misunderstanding.

Addition and subtraction can now be visualized as illustrated in Figs. 320 and 321.

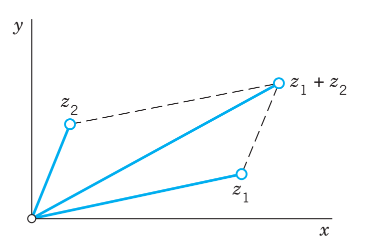


Fig. 320. Addition of complex numbers

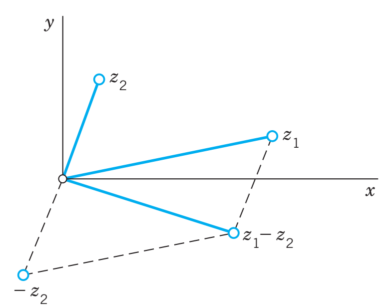


Fig. 321. Subtraction of complex numbers

[2]Sometimes called the **Argand diagram**, after the French mathematician JEAN ROBERT ARGAND (1768–1822), born in Geneva and later librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818), a surveyor of the Danish Academy of Science.

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 322 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

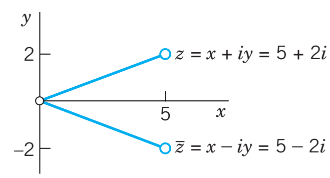


Fig. 322. Complex conjugate numbers

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

(8) $$\text{Re}\, z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \text{Im}\, z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

(9) $$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\text{Im}\, z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$
$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

PROBLEM SET 13.1

1. **Powers of *i*.** Show that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \cdots$ and $1/i = -i, 1/i^2 = -1, 1/i^3 = i, \cdots$.
2. **Rotation.** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 1 + i, z = -1 + 2i, z = 4 - 3i$.
3. **Division.** Verify the calculation in (7). Apply (7) to $(26 - 18i)/(6 - 2i)$.

4. Law for conjugates. Verify (9) for $z_1 = -11 + 10i$, $z_2 = -1 + 4i$.

5. Pure imaginary number. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

6. Multiplication. If the product of two complex numbers is zero, show that at least one factor must be zero.

7. Laws of addition and multiplication. Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \qquad \text{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad \text{(Associative laws)}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \text{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

8–15 COMPLEX ARITHMETIC

Let $z_1 = -2 + 11i$, $z_2 = 2 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$

9. $\text{Re}\,(z_1^2)$, $(\text{Re}\, z_1)^2$

10. $\text{Re}\,(1/z_2^2)$, $1/\text{Re}\,(z_2^2)$

11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$

12. z_1/z_2, z_2/z_1

13. $(z_1 + z_2)(z_1 - z_2)$, $z_1^2 - z_2^2$

14. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$

15. $4\,(z_1 + z_2)/(z_1 - z_2)$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}\,(1/z)$, $\text{Im}\,(1/z^2)$

17. $\text{Re}\, z^4 - (\text{Re}\, z^2)^2$

18. $\text{Re}\,[(1 + i)^{16} z^2]$

19. $\text{Re}\,(z/\bar{z})$, $\text{Im}\,(z/\bar{z})$

20. $\text{Im}\,(1/\bar{z}^2)$

13.2 Polar Form of Complex Numbers. Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the *xy*-coordinates in the complex plane, we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 323). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 324).

θ is called the **argument** of z and is denoted by $\arg z$. Thus $\theta = \arg z$ and (Fig. 323)

$$\tan\theta = \frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 323. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense***.

For $z = 0$ this angle θ is undefined. (Why?) For a given $z \neq 0$ it is determined only up to integer multiples of 2π since cosine and sine are periodic with period 2π. But one often wants to specify a unique value of arg z of a given $z \neq 0$. For this reason one defines the **principal value** Arg z (with capital A!) of arg z by the double inequality

(5) $$-\pi < \operatorname{Arg} z \leqq \pi.$$

Then we have Arg $z = 0$ for positive real $z = x$, which is practical, and Arg $z = \pi$ (not $-\pi$!) for negative real z, e.g., for $z = -4$. The principal value (5) will be important in connection with roots, the complex logarithm (Sec. 13.7), and certain integrals. Obviously, for a given $z \neq 0$, the other values of arg z are arg $z = \operatorname{Arg} z \pm 2n\pi$ $(n = \pm 1, \pm 2, \cdots)$.

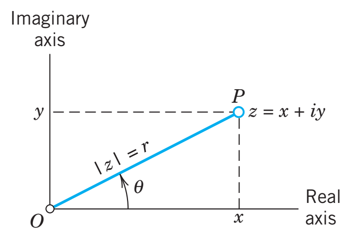


Fig. 323. Complex plane, polar form of a complex number

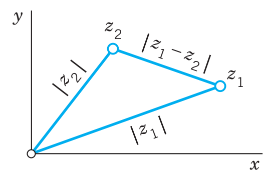


Fig. 324. Distance between two points in the complex plane

EXAMPLE 1 Polar Form of Complex Numbers. Principal Value Arg z

$z = 1 + i$ (Fig. 325) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \quad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \quad \text{(the principal value)}.$$

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and Arg $z = \frac{1}{3}\pi$. ■

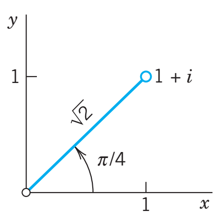


Fig. 325. Example 1

CAUTION! In using (4), we must pay attention to the quadrant in which z lies, since $\tan \theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan \theta_1 = \tan \theta_2 = 1$.

Triangle Inequality

Inequalities such as $x_1 < x_2$ make sense for *real* numbers, but not in complex because *there is no natural way of ordering complex numbers*. However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$ (meaning that z_1 is closer to the origin than z_2) are of great importance. The daily bread of the complex analyst is the **triangle inequality**

(6) $$|z_1 + z_2| \leqq |z_1| + |z_2| \qquad \text{(Fig. 326)}$$

which we shall use quite frequently. This inequality follows by noting that the three points 0, z_1, and $z_1 + z_2$ are the vertices of a triangle (Fig. 326) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 33). (The triangle degenerates if z_1 and z_2 lie on the same straight line through the origin.)

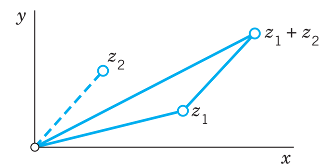


Fig. 326. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6*}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

EXAMPLE 2 Triangle Inequality

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ■

Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Multiplication. By (3) in Sec. 13.1 the product is at first

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{7}$$

Taking absolute values on both sides of (7), we see that *the absolute value of a product equals the* ***product*** *of the absolute values of the factors*,

$$|z_1z_2| = |z_1||z_2|. \tag{8}$$

Taking arguments in (7) shows that *the argument of a product equals the* ***sum*** *of the arguments of the factors*,

$$\arg(z_1z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

Division. We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{10}$$

Similarly, $\arg z_1 = \arg [(z_1/z_2)z_2] = \arg (z_1/z_2) + \arg z_2$ and by subtraction of $\arg z_2$

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

Combining (10) and (11) we also have the analog of (7),

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]. \tag{12}$$

To comprehend this formula, note that it is the polar form of a complex number of absolute value r_1/r_2 and argument $\theta_1 - \theta_2$. But these are the absolute value and argument of z_1/z_2, as we can see from (10), (11), and the polar forms of z_1 and z_2.

EXAMPLE 3 Illustration of Formulas (8)–(11)

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = \frac{2}{3} + (\frac{2}{3})i$. Hence (make a sketch)

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg}(z_1 z_2) = -\frac{3\pi}{4} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi, \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

■

EXAMPLE 4 Integer Powers of *z*. De Moivre's Formula

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$z^n = r^n (\cos n\theta + i \sin n\theta). \tag{13}$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[3]

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{13*}$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos \theta$ and $\sin \theta$. For instance, for $n = 2$ we have on the left $\cos^2 \theta + 2i \cos \theta \sin \theta - \sin^2 \theta$. Taking the real and imaginary parts on both sides of (13*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \qquad \sin 2\theta = 2 \cos \theta \sin \theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$. ■

Roots

If $z = w^n$ $(n = 1, 2, \cdots)$, then to each value of w there corresponds *one* value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an ***n*th root** of z, and we write

[3] ABRAHAM DE MOIVRE (1667–1754), French mathematician, who pioneered the use of complex numbers in trigonometry and also contributed to probability theory (see Sec. 24.8).

(14) $$w = \sqrt[n]{z}.$$

Hence this symbol is ***multivalued***, namely, *n-valued.* The n values of $\sqrt[n]{z}$ can be obtained as follows. We write z and w in polar form

$$z = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad w = R(\cos\phi + i\sin\phi).$$

Then the equation $w^n = z$ becomes, by De Moivre's formula (with ϕ instead of θ),

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta).$$

The absolute values on both sides must be equal; thus, $R^n = r$, so that $R = \sqrt[n]{r}$, where $\sqrt[n]{r}$ is positive real (an absolute value must be nonnegative!) and thus uniquely determined. Equating the arguments $n\phi$ and θ and recalling that θ is determined only up to integer multiples of 2π, we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n-1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$, hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

(15) $$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n-1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

(16) $$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n-1.$$

These n values are called the ***n*****th roots of unity**. They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 327–329 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

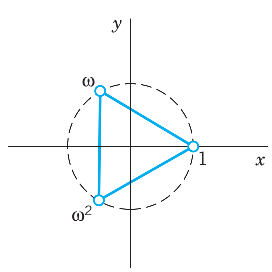


Fig. 327. $\sqrt[3]{1}$

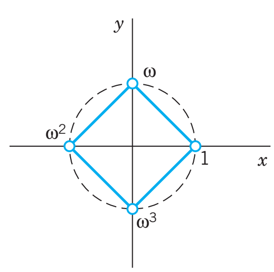


Fig. 328. $\sqrt[4]{1}$

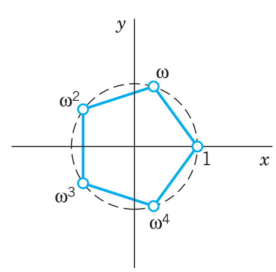


Fig. 329. $\sqrt[5]{1}$

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number $z\ (\neq 0)$, then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

PROBLEM SET 13.2

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$

2. $-4 + 4i$

3. $2i, \quad -2i$

4. -5

5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$

6. $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$

7. $1 + \frac{1}{2}\pi i$

8. $\dfrac{-4 + 19i}{2 + 5i}$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$

10. $-5, \quad -5 - i, \quad -5 + i$

11. $3 \pm 4i$

12. $-\pi - \pi i$

13. $(1 + i)^{20}$

14. $-1 + 0.1i, \quad -1 - 0.1i$

15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $3\,(\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi)$

16. $6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$

17. $\sqrt{8}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$

18. $\sqrt{50}\,(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$

ROOTS

19. CAS PROJECT. Roots of Unity and Their Graphs. Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

20. TEAM PROJECT. Square Root. (a) Show that $w = \sqrt{z}$ has the values

$$\begin{aligned} w_1 &= \sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right], \\ w_2 &= \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] \\ &= -w_1. \end{aligned} \tag{18}$$

(b) Obtain from (18) the often more practical formula

$$\sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\operatorname{sign} y)i\sqrt{\tfrac{1}{2}(|z| + x)}\right] \tag{19}$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $-14i$, $-9 - 40i$, and $1 + \sqrt{48}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 + i}$

22. $\sqrt[3]{3 + 4i}$

23. $\sqrt[3]{216}$

24. $\sqrt[4]{-4}$

25. $\sqrt[4]{i}$

26. $\sqrt[8]{1}$

27. $\sqrt[5]{-1}$

28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 + z + 1 - i = 0$

30. $z^4 + 324 = 0$. Using the solutions, factor $z^4 + 324$ into quadratic factors with *real* coefficients.

31. $z^4 - 6iz^2 + 16 = 0$

32–35 INEQUALITIES AND EQUALITY

32. Triangle inequality. Verify (6) for $z_1 = 3 + i$, $z_2 = -2 + 4i$

33. Triangle inequality. Prove (6).

34. Re and Im. Prove $|\text{Re}\, z| \leqq |z|$, $|\text{Im}\, z| \leqq |z|$.

35. Parallelogram equality. Prove and explain the name

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2).$$

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or *different* from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

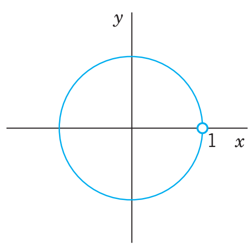


Fig. 330. Unit circle

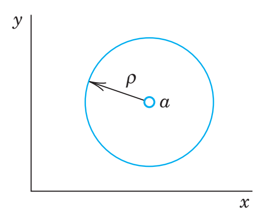


Fig. 331. Circle in the complex plane

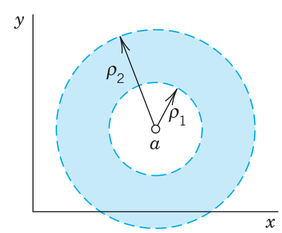


Fig. 332. Annulus in the complex plane

because it is the set of all z whose distance $|z - a|$ from the center a equals ρ. Accordingly, its interior ("**open circular disk**") is given by $|z - a| < \rho$, its interior plus the circle itself ("**closed circular disk**") by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of a or, more precisely, a *ρ-neighborhood* of a. And a has infinitely many of them, one for each value of ρ (> 0), and a is a point of each of them, by definition!

In modern literature *any set* containing a ρ-neighborhood of a is also called a *neighborhood* of a.

Figure 332 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all z whose distance $|z - a|$ from a is greater than ρ_1 but less than ρ_2. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

Half-Planes. By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

For Reference: Concepts on Sets in the Complex Plane

To our discussion of special sets let us add some general concepts related to sets that we shall need throughout Chaps. 13–18; keep in mind that you can find them here.

By a **point set** in the complex plane we mean any sort of collection of finitely many or infinitely many points. Examples are the solutions of a quadratic equation, the points of a line, the points in the interior of a circle as well as the sets discussed just before.

A set S is called **open** if every point of S has a neighborhood consisting entirely of points that belong to S. For example, the points in the interior of a circle or a square form an open set, and so do the points of the right half-plane Re $z = x > 0$.

A set S is called **connected** if any two of its points can be joined by a chain of finitely many straight-line segments all of whose points belong to S. An open and connected set is called a **domain.** Thus an open disk and an open annulus are domains. An open square with a diagonal removed is not a domain since this set is not connected. (Why?)

The **complement** of a set S in the complex plane is the set of all points of the complex plane that ***do not belong*** to S. A set S is called **closed** if its complement is open. For example, the points on and inside the unit circle form a closed set ("closed unit disk") since its complement $|z| > 1$ is open.

A **boundary point** of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S. For example, the boundary points of an annulus are the points on the two bounding circles. Clearly, if a set S is open, then no boundary point belongs to S; if S is closed, then every boundary point belongs to S. The set of all boundary points of a set S is called the **boundary** of S.

A **region** is a set consisting of a domain plus, perhaps, some or all of its boundary points. WARNING! "Domain" is the *modern* term for an open connected set. Nevertheless, some authors still call a domain a "region" and others make no distinction between the two terms.

Complex Function

Complex analysis is concerned with complex functions that are differentiable in some domain. Hence we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be similar to that in calculus. Nevertheless it needs great attention because it will show interesting basic differences between real and complex calculus.

Recall from calculus that a *real* function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number $f(x)$, called the *value* of f at x. Now in complex, S is a set of *complex* numbers. And a **function** f defined on S is a rule that assigns to every z in S a complex number w, called the *value* of f at z. We write

$$w = f(z).$$

Here z varies in S and is called a **complex variable**. The set S is called the *domain of definition* of f or, briefly, the **domain** of f. (In most cases S will be open and connected, thus a domain as defined just before.)

Example: $w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the **range** *of* f.

w is complex, and we write $w = u + iv$, where u and v are the real and imaginary parts, respectively. Now w depends on $z = x + iy$. Hence u becomes a real function of x and y, and so does v. We may thus write

$$w = f(z) = u(x, y) + iv(x, y).$$

This shows that a *complex* function $f(z)$ is equivalent to a *pair* of *real* functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y.

EXAMPLE 1 **Function of a Complex Variable**

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Solution. $u = \text{Re}\, f(z) = x^2 - y^2 + 3x$ and $v = 2xy + 3y$. Also,

$$f(1 + 3i) = (1 + 3i)^2 + 3(1 + 3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i.$$

This shows that $u(1, 3) = -5$ and $v(1, 3) = 15$. Check this by using the expressions for u and v. ■

EXAMPLE 2 **Function of a Complex Variable**

Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$.

Solution. $f(z) = 2i(x + iy) + 6(x - iy)$ gives $u(x, y) = 6x - 2y$ and $v(x, y) = 2x - 6y$. Also,

$$f(\tfrac{1}{2} + 4i) = 2i(\tfrac{1}{2} + 4i) + 6(\tfrac{1}{2} - 4i) = i - 8 + 3 - 24i = -5 - 23i.$$

Check this as in Example 1. ■

Remarks on Notation and Terminology

1. Strictly speaking, $f(z)$ denotes the value of f at z, but it is a convenient abuse of language to talk about *the function* $f(z)$ (instead of *the function* f), thereby exhibiting the notation for the independent variable.

2. We assume all functions to be ***single-valued relations***, as usual: to each z in S there corresponds but *one* value $w = f(z)$ (but, of course, several z may give the same value $w = f(z)$, just as in calculus). Accordingly, we shall *not use* the term "multivalued function" (used in some books on complex analysis) for a multivalued relation, in which to a z there corresponds more than one w.

Limit, Continuity

A function $f(z)$ is said to have the **limit** l as z approaches a point z_0, written

$$\lim_{z \to z_0} f(z) = l, \tag{1}$$

if f is defined in a neighborhood of z_0 (except perhaps at z_0 itself) and if the values of f are "close" to l for all z "close" to z_0; in precise terms, if for every positive real ϵ we can find a positive real δ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$ (Fig. 333) we have

$$|f(z) - l| < \epsilon; \tag{2}$$

geometrically, if for every $z \neq z_0$ in that δ-disk the value of f lies in the disk (2).

Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real case, x can approach an x_0 only along the real line, here, by definition,

z may approach z_0 ***from any direction*** in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique. (See Team Project 24.)

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{3}$$

Note that by definition of a limit this implies that $f(z)$ is defined in some neighborhood of z_0.

$f(z)$ is said to be *continuous in a domain* if it is continuous at each point of this domain.

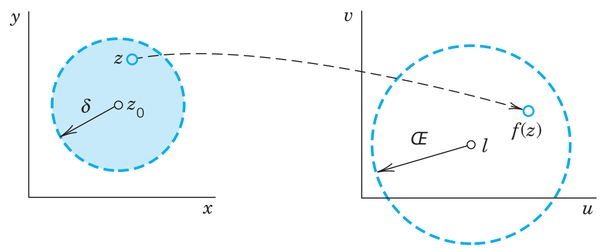


Fig. 333. Limit

Derivative

The **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{4}$$

provided this limit exists. Then f is said to be **differentiable** at z_0. If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and (4) takes the form

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{4'}$$

Now comes an ***important point***. Remember that, by the definition of limit, $f(z)$ is defined in a neighborhood of z_0 and z in (4′) may approach z_0 from any direction in the complex plane. Hence differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in (4′) always approaches a certain value and all these values are equal. This is important and should be kept in mind.

EXAMPLE 3 Differentiability. Derivative

The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$ ■

The ***differentiation rules*** *are the same as in real calculus,* since their proofs are literally the same. Thus for any differentiable functions f and g and constant c we have

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

as well as the chain rule and the power rule $(z^n)' = nz^{n-1}$ (n integer).

Also, if $f(z)$ is differentiable at z_0, it is continuous at z_0. (See Team Project 24.)

EXAMPLE 4 $\bar{z}$ **not Differentiable**

It may come as a surprise that there are many complex functions that do not have a derivative at any point. For instance, $f(z) = \bar{z} = x - iy$ is such a function. To see this, we write $\Delta z = \Delta x + i\,\Delta y$ and obtain

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}. \tag{5}$$

If $\Delta y = 0$, this is $+1$. If $\Delta x = 0$, this is -1. Thus (5) approaches $+1$ along path I in Fig. 334 but -1 along path II. Hence, by definition, the limit of (5) as $\Delta z \to 0$ does not exist at any z. ■

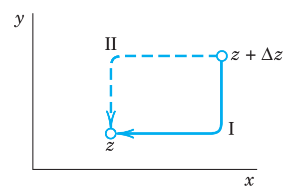


Fig. 334. Paths in (5)

Surprising as Example 4 may be, it merely illustrates that differentiability of a *complex* function is a rather severe requirement.

The idea of proof (approach of z from different directions) is basic and will be used again as the crucial argument in the next section.

Analytic Functions

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Hence analyticity of $f(z)$ at z_0 means that $f(z)$ has a derivative at every point in some neighborhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighborhoods). This concept is *motivated* by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but not throughout some neighborhood of z_0. Team Project 24 gives an example.

A more modern term for *analytic in D* is *holomorphic in D*.

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 5i| \leqq \frac{3}{2}$
2. $0 < |z| < 1$
3. $\pi < |z - 4 + 2i| < 3\pi$
4. $-\pi < \text{Im } z < \pi$
5. $|\arg z| < \frac{1}{4}\pi$
6. $\text{Re } (1/z) < 1$
7. $\text{Re } z \geqq -1$
8. $|z + i| \geqq |z - i|$
9. **WRITING PROJECT. Sets in the Complex Plane.** Write a report by formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find $\text{Re } f$, and $\text{Im } f$ and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$
11. $f(z) = 1/(1 - z)$ at $1 - i$
12. $f(z) = (z - 2)/(z + 2)$ at $8i$
13. **CAS PROJECT. Graphing Functions.** Find and graph $\text{Re } f$, $\text{Im } f$, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves $\text{Re } f(z) = \text{const}$ and $\text{Im } f(z) = \text{const}$ in the same figure, and the curves $|f(z)| = \text{const}$ in another figure, where **(a)** $f(z) = z^2$, **(b)** $f(z) = 1/z$, (c) $f(z) = z^4$.

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\text{Re } z^2)/|z|$
15. $|z|^2 \text{ Im } (1/z)$
16. $(\text{Im } z^2)/|z|^2$
17. $(\text{Re } z)/(1 - |z|)$

18–23 Differentiation. Find the value of the derivative of

18. $(z - i)/(z + i)$ at i
19. $(z - 4i)^8$ at $= 3 + 4i$
20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.
21. $i(1 - z)^n$ at 0
22. $(iz^3 + 3z^2)^3$ at $2i$
23. $z^3/(z + i)^3$ at i
24. **TEAM PROJECT. Limit, Continuity, Derivative**

    (a) Limit. Prove that (1) is equivalent to the pair of relations

    $$\lim_{z \to z_0} \text{Re } f(z) = \text{Re } l, \qquad \lim_{z \to z_0} \text{Im } f(z) = \text{Im } l.$$

    (b) Limit. If $\lim_{z \to z_0} f(x)$ exists, show that this limit is unique.

    (c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re } z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

25. WRITING PROJECT. Comparison with Calculus. Summarize the second part of this section beginning with *Complex Function*, and indicate what is conceptually analogous to calculus and what is not.

13.4 Cauchy–Riemann Equations. Laplace's Equation

As we saw in the last section, to do complex analysis (i.e., "calculus in the complex") on any complex function, we require that function to be *analytic on some domain* that is differentiable in that domain.

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

THEOREM 1

Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) *at all points of D.*

[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GenRef9] in App. 1.

PROOF By assumption, the derivative $f'(z)$ at z exists. It is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

The idea of the proof is very simple. By the definition of a limit in complex (Sec. 13.3), we can let Δz approach zero along any path in a neighborhood of z. Thus we may choose the two paths I and II in Fig. 335 and equate the results. By comparing the real parts we shall obtain the first Cauchy–Riemann equation and by comparing the imaginary parts the second. The technical details are as follows.

We write $\Delta z = \Delta x + i\,\Delta y$. Then $z + \Delta z = x + \Delta x + i(y + \Delta y)$, and in terms of u and v the derivative in (2) becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\,\Delta y}. \tag{3}$$

We first choose path I in Fig. 335. Thus we let $\Delta y \to 0$ first and then $\Delta x \to 0$. After Δy is zero, $\Delta z = \Delta x$. Then (3) becomes, if we first write the two u-terms and then the two v-terms,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}.$$

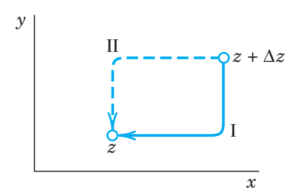


Fig. 335. Paths in (2)

Since $f'(z)$ exists, the two real limits on the right exist. By definition, they are the partial derivatives of u and v with respect to x. Hence the derivative $f'(z)$ of $f(z)$ can be written

$$f'(z) = u_x + iv_x. \tag{4}$$

Similarly, if we choose path II in Fig. 335, we let $\Delta x \to 0$ first and then $\Delta y \to 0$. After Δx is zero, $\Delta z = i\,\Delta y$, so that from (3) we now obtain

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\,\Delta y}.$$

Since $f'(z)$ exists, the limits on the right exist and give the partial derivatives of u and v with respect to y; noting that $1/i = -i$, we thus obtain

$$f'(z) = -iu_y + v_y. \tag{5}$$

The existence of the derivative $f'(z)$ thus implies the existence of the four partial derivatives in (4) and (5). By equating the real parts u_x and v_y in (4) and (5) we obtain the first

Cauchy–Riemann equation (1). Equating the imaginary parts gives the other. This proves the first statement of the theorem and implies the second because of the definition of analyticity. ■

Formulas (4) and (5) are also quite practical for calculating derivatives $f'(z)$, as we shall see.

EXAMPLE 1 **Cauchy–Riemann Equations**

$f(z) = z^2$ is analytic for all z. It follows that the Cauchy–Riemann equations must be satisfied (as we have verified above).

For $f(z) = \bar{z} = x - iy$ we have $u = x, v = -y$ and see that the second Cauchy–Riemann equation is satisfied, $u_y = -v_x = 0$, but the first is not: $u_x = 1 \neq v_y = -1$. We conclude that $f(z) = \bar{z}$ is not analytic, confirming Example 4 of Sec. 13.3. Note the savings in calculation! ■

The Cauchy–Riemann equations are fundamental because they are not only necessary but also sufficient for a function to be analytic. More precisely, the following theorem holds.

THEOREM 2 **Cauchy–Riemann Equations**

If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have **continuous** *first partial derivatives that satisfy the Cauchy–Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.*

The proof is more involved than that of Theorem 1 and we leave it optional (see App. 4).

Theorems 1 and 2 are of great practical importance, since, by using the Cauchy–Riemann equations, we can now easily find out whether or not a given complex function is analytic.

EXAMPLE 2 **Cauchy–Riemann Equations. Exponential Function**

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

Solution. We have $u = e^x \cos y, v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$

$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

We see that the Cauchy–Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z. ($f(z)$ will be the complex analog of e^x known from calculus.) ■

EXAMPLE 3 **An Analytic Function of Constant Absolute Value Is Constant**

The Cauchy–Riemann equations also help in deriving general properties of analytic functions.

For instance, show that if $f(z)$ is analytic in a domain D and $|f(z)| = k = \text{const}$ in D, then $f(z) = \text{const}$ in D. (We shall make crucial use of this in Sec. 18.6 in the proof of Theorem 3.)

Solution. By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2$. By differentiation,

$$uu_x + vv_x = 0,$$

$$uu_y + vv_y = 0.$$

Now use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, to get

$$\text{(6)} \qquad \begin{aligned} &\text{(a)} \quad uu_x - vu_y = 0, \\ &\text{(b)} \quad uu_y - vu_x = 0. \end{aligned}$$

To get rid of u_y, multiply (6a) by u and (6b) by v and add. Similarly, to eliminate u_x, multiply (6a) by $-v$ and (6b) by u and add. This yields

$$(u^2 + v^2)u_x = 0,$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $v_x = v_y = 0$. Together this implies $u = \text{const}$ and $v = \text{const}$; hence $f = \text{const}$. ■

We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 1)

$$u_r = \frac{1}{r}v_\theta, \qquad v_r = -\frac{1}{r}u_\theta \qquad (r > 0). \tag{7}$$

Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat conduction, and other applications (see Chaps. 12 and 18).

THEOREM 3

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{8}$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0, \tag{9}$$

in D and have continuous second partial derivatives in D.

PROOF Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we have

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}. \tag{10}$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$ ■

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens a wide range of engineering and physical applications, as shown in Chap. 18.

PROBLEM SET 13.4

1. **Cauchy–Riemann equations in polar form.** Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$
3. $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$
4. $f(z) = e^x(\cos y - i \sin y)$
5. $f(z) = \text{Re}(z^2) - i \,\text{Im}(z^2)$
6. $f(z) = 1/(z - z^5)$
7. $f(z) = i/z^8$
8. $f(z) = \text{Arg}\, 2\pi z$
9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$
10. $f(z) = \ln |z| + i \,\text{Arg}\, z$
11. $f(z) = \cos x \cosh y - i \sin x \sinh y$

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = x^2 + y^2$
13. $u = xy$
14. $v = xy$
15. $u = x/(x^2 + y^2)$
16. $u = \sin x \cosh y$
17. $v = (2x + 1)y$
18. $u = x^3 - 3xy^2$
19. $v = e^x \sin 2y$
20. **Laplace's equation.** Give the details of the derivative of (9).

21–24

Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{\pi x} \cos av$
22. $u = \cos ax \cosh 2y$
23. $u = ax^3 + bxy$
24. $u = \cosh ax \cos y$
25. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = $ const of a harmonic function u and of its conjugate v on the same axes. Apply the program to **(a)** $u = x^2 - y^2$, $v = 2xy$, **(b)** $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.
26. Apply the program in Prob. 25 to $u = e^x \cos y$, $v = e^x \sin y$ and to an example of your own.

27. Harmonic conjugate. Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

28. Illustrate Prob. 27 by an example.

29. Two further formulas for the derivative. Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

30. TEAM PROJECT. Conditions for $f(z) = \text{const}$. Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = \text{const}$

(a) $\text{Re}\, f(z) = \text{const}$

(b) $\text{Im}\, f(z) = \text{const}$

(c) $f'(z) = 0$

(d) $|f(z)| = \text{const}$ (see Example 3)

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$\textbf{(1)} \qquad e^z = e^x(\cos y + i \sin y).$$

This definition is motivated by the fact the e^z ***extends*** the real exponential function e^x of calculus in a natural fashion. Namely:

(A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

(B) e^z is analytic for all z. (Proved in Example 2 of Sec. 13.4.)

(C) The derivative of e^z is e^z, that is,

$$(2) \qquad (e^z)' = e^z.$$

This follows from (4) in Sec. 13.4,

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

REMARK. This definition provides for a relatively simple discussion. We could define e^z by the familiar series $1 + x + x^2/2! + x^3/3! + \cdots$ with x replaced by z, but we would then have to discuss complex series at this very early stage. (We will show the connection in Sec. 15.4.)

Further Properties. A function $f(z)$ that is analytic for all z is called an **entire function**. Thus, e^z is entire. Just as in calculus the ***functional relation***

$$(3) \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}$$

holds for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Indeed, by (1),

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2).$$

Since $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for these *real* functions, by an application of the addition formulas for the cosine and sine functions (similar to that in Sec. 13.2) we see that

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}$$

as asserted. An interesting special case of (3) is $z_1 = x$, $z_2 = iy$; then

$$e^z = e^x e^{iy}. \tag{4}$$

Furthermore, for $z = iy$ we have from (1) the so-called **Euler formula**

$$e^{iy} = \cos y + i \sin y. \tag{5}$$

Hence the **polar form** of a complex number, $z = r(\cos\theta + i \sin\theta)$, may now be written

$$z = re^{i\theta}. \tag{6}$$

From (5) we obtain

$$e^{2\pi i} = 1 \tag{7}$$

as well as the important formulas (verify!)

$$e^{\pi i/2} = i, \qquad e^{\pi i} = -1, \qquad e^{-\pi i/2} = -i, \qquad e^{-\pi i} = -1. \tag{8}$$

Another consequence of (5) is

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1. \tag{9}$$

That is, for pure imaginary exponents, the exponential function has absolute value 1, a result you should remember. From (9) and (1),

$$|e^z| = e^x. \qquad \text{Hence} \qquad \arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots), \tag{10}$$

since $|e^z| = e^x$ shows that (1) is actually e^z in polar form.

From $|e^z| = e^x \neq 0$ in (10) we see that

$$e^x \neq 0 \qquad \text{for all } z. \tag{11}$$

So here we have an entire function that never vanishes, in contrast to (nonconstant) polynomials, which are also entire (Example 5 in Sec. 13.3) but always have a zero, as is proved in algebra.

Periodicity of e^x with period $2\pi i$,

$$e^{z+2\pi i} = e^z \qquad \text{for all } z \tag{12}$$

is a basic property that follows from (1) and the periodicity of cos y and sin y. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π

$$-\pi < y \leqq \pi \qquad \text{(Fig. 336).} \tag{13}$$

This infinite strip is called a **fundamental region** of e^z.

EXAMPLE 1 Function Values. Solution of Equations

Computation of values from (1) provides no problem. For instance,

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-1.6i}| = e^{1.4} = 4.055, \qquad \text{Arg } e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

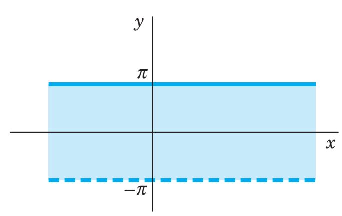


Fig. 336. Fundamental region of the exponential function e^z in the z-plane

PROBLEM SET 13.5

1. e^z is entire. Prove this.

2–7 Function Values. Find e^z in the form $u + iv$ and $|e^z|$ if z equals

2. $3 + 4i$

3. $2\pi i(1 + i)$

4. $0.6 - 1.8i$

5. $2 + 3\pi i$

6. $11\pi i/2$

7. $\sqrt{2} + \frac{1}{2}\pi i$

8–13 Polar Form. Write in exponential form (6):

8. $\sqrt[n]{z}$

9. $4 + 3i$

10. $\sqrt{i}$, $\sqrt{-i}$

11. -6.3

12. $1/(1 - z)$

13. $1 + i$

14–17 Real and Imaginary Parts. Find Re and Im of

14. $e^{-\pi z}$

15. $\exp(z^2)$

16. $e^{1/z}$ **17.** $\exp(z^3)$

18. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos (x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

19–22 Equations. Find all solutions and graph some of them in the complex plane.

19. $e^z = 1$ **20.** $e^z = 4 + 3i$

21. $e^z = 0$ **22.** $e^z = -2$

13.6 Trigonometric and Hyperbolic Functions. Euler's Formula

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions.* We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\textbf{(1)} \qquad \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$(2) \qquad \tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

and

$$(3) \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Since e^z is entire, $\cos z$ and $\sin z$ are entire functions. $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero; and $\cot z$ and $\csc z$ are analytic except where $\sin z$ is zero. Formulas for the derivatives follow readily from $(e^z)' = e^z$ and (1)–(3); as in calculus,

$$(4) \qquad (\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z,$$

etc. Equation (1) also shows that **Euler's formula *is valid in complex***:

$$\textbf{(5)} \qquad e^{iz} = \cos z + i \sin z \qquad \text{for all } z.$$

The real and imaginary parts of $\cos z$ and $\sin z$ are needed in computing values, and they also help in displaying properties of our functions. We illustrate this with a typical example.

EXAMPLE 1 Real and Imaginary Parts. Absolute Value. Periodicity

Show that

$$\textbf{(6)} \qquad \begin{aligned} &\textbf{(a)} \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &\textbf{(b)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

and

$$(7) \qquad \begin{aligned} &\textbf{(a)} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &\textbf{(b)} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y \end{aligned}$$

and give some applications of these formulas.

Solution. From (1),

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \tfrac{1}{2}(e^y + e^{-y}) \cos x - \tfrac{1}{2}i(e^y - e^{-y}) \sin x. \end{aligned}$$

This yields (6a) since, as is known from calculus,

$$(8) \qquad \cosh y = \tfrac{1}{2}(e^y + e^{-y}), \qquad \sinh y = \tfrac{1}{2}(e^y - e^{-y});$$

(6b) is obtained similarly. From (6a) and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2 + 3i) = \cos 2 \cosh 3 - i \sin 2 \sinh 3 = -4.190 - 9.109i$.

From (6) we see that $\sin z$ and $\cos z$ are ***periodic with period 2π***, just as in real. Periodicity of $\tan z$ and $\cot z$ with period π now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine; whereas $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$, the complex cosine and sine functions are ***no longer bounded*** but approach infinity in absolute value as $y \to \infty$, since then $\sinh y \to \infty$ in (7). ■

EXAMPLE 2 Solutions of Equations. Zeros of cos z and sin z

Solve (a) $\cos z = 5$ (which has no real solution!), (b) $\cos z = 0$, (c) $\sin z = 0$.

Solution. (a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solutions (rounded off to 3 decimals)

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \quad \text{and} \quad 0.101.$$

Thus $e^{-y} = 9.899$ or 0.101, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$. *Ans.* $z = \pm 2n\pi \pm 2.292i$ $(n = 0, 1, 2, \cdots)$.

Can you obtain this from (6a)?

(b) $\cos x = 0$, $\sinh y = 0$ by (7a), $y = 0$. *Ans.* $z = \pm\frac{1}{2}(2n + 1)\pi$ $(n = 0, 1, 2, \cdots)$.
(c) $\sin x = 0$, $\sinh y = 0$ by (7b), *Ans.* $z = \pm n\pi$ $(n = 0, 1, 2, \cdots)$.
Hence the only zeros of $\cos z$ and $\sin z$ are those of the real cosine and sine functions. ■

General formulas *for the real trigonometric functions continue to hold for complex values.* This follows immediately from the definitions. We mention in particular the addition rules

$$\begin{aligned}\cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2\\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1\end{aligned} \tag{9}$$

and the formula

$$\cos^2 z + \sin^2 z = 1. \tag{10}$$

Some further useful formulas are included in the problem set.

Hyperbolic Functions

The complex **hyperbolic cosine** and **sine** are defined by the formulas

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}). \tag{11}$$

This is suggested by the familiar definitions for a real variable [see (8)]. These functions are entire, with derivatives

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z, \tag{12}$$

as in calculus. The other hyperbolic functions are defined by

$$\begin{aligned}\tanh z &= \frac{\sinh z}{\cosh z}, &\qquad \coth z &= \frac{\cosh z}{\sinh z},\\ \operatorname{sech} z &= \frac{1}{\cosh z}, &\qquad \operatorname{csch} z &= \frac{1}{\sinh z}.\end{aligned} \tag{13}$$

Complex Trigonometric and Hyperbolic Functions Are Related. If in (11), we replace z by iz and then use (1), we obtain

$$\cosh iz = \cos z, \qquad \sinh iz = i \sin z. \tag{14}$$

Similarly, if in (1) we replace z by iz and then use (11), we obtain conversely

$$\cos iz = \cosh z, \qquad \sin iz = i \sinh z. \tag{15}$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

PROBLEM SET 13.6

1–4 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

1. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

   $\sinh z = \sinh x \cos y + i \cosh x \sin y.$

2. $\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

   $\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

3. $\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$
4. **Entire Functions.** Prove that $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are entire.
5. **Harmonic Functions.** Verify by differentiation that $\operatorname{Im} \cos z$ and $\operatorname{Re} \sin z$ are harmonic.

6–12 Function Values. Find, in the form $u + iv$,

6. $\sin 2\pi i$
7. $\cos i, \quad \sin i$
8. $\cos \pi i, \quad \cosh \pi i$
9. $\cosh(-1 + 2i), \quad \cos(-2 - i)$
10. $\sinh(3 + 4i), \quad \cosh(3 + 4i)$
11. $\sin \pi i, \quad \cos(\frac{1}{2}\pi - \pi i)$
12. $\cos \frac{1}{2}\pi i, \quad \cos[\frac{1}{2}\pi(1 + i)]$

13–15 Equations and Inequalities. Using the definitions, prove:

13. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.
14. $|\sinh y| \leqq |\cos z| \leqq \cosh y, |\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.
15. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin(z_1 + z_2) + \sin(z_1 - z_2)]$

16–19 Equations. Find all solutions.

16. $\sin z = 100$
17. $\cosh z = 0$
18. $\cosh z = -1$
19. $\sinh z = 0$
20. **Re tan z and Im tan z.** Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

13.7 Logarithm. General Power. Principal Value

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now, from Sec. 13.5, we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z). \tag{1}$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm* $\ln z$ ($z \neq 0$) *is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\operatorname{Arg} z$ (see Sec. 13.2) is denoted by $\operatorname{Ln} z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z \qquad (z \neq 0). \tag{2}$$

The uniqueness of $\operatorname{Arg} z$ for given z ($\neq 0$) implies that $\operatorname{Ln} z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of 2π, the other values of $\ln z$ are given by

$$\ln z = \operatorname{Ln} z \pm 2n\pi i \qquad (n = 1, 2, \cdots). \tag{3}$$

They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

If z is positive real, then $\operatorname{Arg} z = 0$, and $\operatorname{Ln} z$ becomes identical with the real natural logarithm known from calculus. If z is negative real (so that the natural logarithm of calculus is not defined!), then $\operatorname{Arg} z = \pi$ and

$$\operatorname{Ln} z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg(e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln(e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

EXAMPLE 1 Natural Logarithm. Principal Value

$$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \cdots \qquad \operatorname{Ln} 1 = 0$$

$$\ln 4 = 1.386294 \pm 2n\pi i \qquad \operatorname{Ln} 4 = 1.386294$$

$$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots \qquad \operatorname{Ln}(-1) = \pi i$$

$$\ln(-4) = 1.386294 \pm (2n + 1)\pi i \qquad \operatorname{Ln}(-4) = 1.386294 + \pi i$$

$$\ln i = \pi i/2, -3\pi/2, 5\pi i/2, \cdots \qquad \operatorname{Ln} i = \pi i/2$$

$$\ln 4i = 1.386294 + \pi i/2 \pm 2n\pi i \qquad \operatorname{Ln} 4i = 1.386294 + \pi i/2$$

$$\ln(-4i) = 1.386294 - \pi i/2 \pm 2n\pi i \qquad \operatorname{Ln}(-4i) = 1.386294 - \pi i/2$$

$$\ln(3 - 4i) = \ln 5 + i \arg(3 - 4i) \qquad \operatorname{Ln}(3 - 4i) = 1.609438 - 0.927295i$$

$$= 1.609438 - 0.927295i \pm 2n\pi i \qquad \text{(Fig. 337)}$$

■

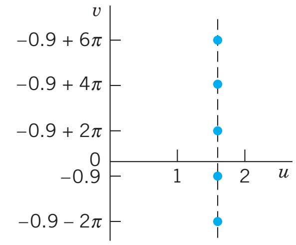


Fig. 337. Some values of ln (3 − 4*i*) in Example 1

The familiar relations for the natural logarithm continue to hold for complex values, that is,

$$\text{(5)} \qquad \text{(a)} \quad \ln (z_1 z_2) = \ln z_1 + \ln z_2, \qquad \text{(b)} \quad \ln (z_1/z_2) = \ln z_1 - \ln z_2$$

but these relations are to be understood in the sense that each value of one side is also contained among the values of the other side; see the next example.

EXAMPLE 2 **Illustration of the Functional Relation (5) in Complex**

Let

$$z_1 = z_2 = e^{\pi i} = -1.$$

If we take the principal values

$$\text{Ln } z_1 = \text{Ln } z_2 = \pi i,$$

then (5a) holds provided we write $\ln (z_1 z_2) = \ln 1 = 2\pi i$; however, it is not true for the principal value, $\text{Ln}(z_1 z_2) = \text{Ln } 1 = 0$. ■

THEOREM 1

Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ *formula* (3) *defines a function, which is analytic, except at* 0 *and on the negative real axis, and has the derivative*

$$\text{(6)} \qquad (\ln z)' = \frac{1}{z} \qquad (z \text{ not 0 or negative real}).$$

PROOF We show that the Cauchy–Riemann equations are satisfied. From (1)–(3) we have

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2} \ln (x^2 + y^2) + i\left(\arctan \frac{y}{x} + c\right)$$

where the constant c is a multiple of 2π. By differentiation,

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold. [Confirm this by using these equations in polar form, which we did not use since we proved them only in the problems (to Sec. 13.4).] Formula (4) in Sec. 13.4 now gives (6),

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 338. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

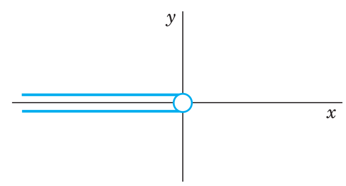


Fig. 338. Branch cut for ln z

General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

$$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0). \tag{7}$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* z^c.

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the n distinct values of the nth root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and z^c has only finitely many distinct values. However, if c is real irrational or genuinely complex, then z^c is infinitely many-valued.

EXAMPLE 3 General Power

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right] = e^{-(\pi/2)\mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$\begin{aligned}(1 + i)^{2-i} &= \exp[(2 - i)\ln(1 + i)] = \exp[(2 - i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}]\\ &= 2e^{\pi/4 \pm 2n\pi}[\sin(\tfrac{1}{2}\ln 2) + i\cos(\tfrac{1}{2}\ln 2)].\end{aligned}$$

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\operatorname{Ln} z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–4 VERIFICATIONS IN THE TEXT

1. Verify the computations in Example 1.
2. Verify (5) for $z_1 = -i$ and $z_2 = -1$.
3. Prove analyticity of $\operatorname{Ln} z$ by means of the Cauchy–Riemann equations in polar form (Sec. 13.4).
4. Prove (4a) and (4b).

COMPLEX NATURAL LOGARITHM *ln z*

5–11 Principal Value Ln *z*. Find $\operatorname{Ln} z$ when z equals

5. -11
6. $4 + 4i$
7. $4 - 4i$
8. $1 \pm i$
9. $0.6 + 0.8i$
10. $-15 \pm 0.1i$
11. ei

12–16 All Values of ln *z*. Find all values and graph some of them in the complex plane.

12. $\ln e$
13. $\ln 1$
14. $\ln(-7)$
15. $\ln(e^i)$
16. $\ln(4 + 3i)$
17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z.

18. $\ln z = -\pi i/2$
19. $\ln z = 4 - 3i$
20. $\ln z = e - \pi i$
21. $\ln z = 0.6 + 0.4i$

22–28 General Powers. Find the principal value. Show details.

22. $(2i)^{2i}$
23. $(1 + i)^{1-i}$
24. $(1 - i)^{1+i}$
25. $(-3)^{3-i}$
26. $(i)^{i/2}$
27. $(-1)^{2-i}$
28. $(3 + 4i)^{1/3}$
29. How can you find the answer to Prob. 24 from the answer to Prob. 23?
30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent**, **inverse cotangent**, **inverse hyperbolic sine**, etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued***.) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$

(b) $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$

(c) $\operatorname{arccosh} z = \ln(z + \sqrt{z^2 - 1})$

(d) $\operatorname{arcsinh} z = \ln(z + \sqrt{z^2 + 1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Divide $15 + 23i$ by $-3 + 7i$. Check the result by multiplication.

2. What happens to a quotient if you take the complex conjugates of the two numbers? If you take the absolute values of the numbers?

3. Write the two numbers in Prob. 1 in polar form. Find the principal values of their arguments.

4. State the definition of the derivative from memory. Explain the big difference from that in calculus.

5. What is an analytic function of a complex variable?

6. Can a function be differentiable at a point without being analytic there? If yes, give an example.

7. State the Cauchy–Riemann equations. Why are they of basic importance?

8. Discuss how e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are related.

9. $\ln z$ is more complicated than $\ln x$. Explain. Give examples.

10. How are general powers defined? Give an example. Convert it to the form $x + iy$.

11–16 **Complex Numbers.** Find, in the form $x + iy$, showing details,

11. $(2 + 3i)^2$ **12.** $(1 - i)^{10}$

13. $1/(4 + 3i)$ **14.** $\sqrt{i}$

15. $(1 + i)/(1 - i)$ **16.** $e^{\pi i/2}$, $e^{-\pi i/2}$

17–20 **Polar Form.** Represent in polar form, with the principal argument.

17. $-4 - 4i$ **18.** $12 + i$, $12 - i$

19. $-15i$ **20.** $0.6 + 0.8i$

21–24 **Roots.** Find and graph all values of:

21. $\sqrt{81}$ **22.** $\sqrt{-32i}$

23. $\sqrt[4]{-1}$ **24.** $\sqrt[3]{1}$

25–30 **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check by the Cauchy–Riemann equations for analyticity.

25 $u = xy$ **26.** $v = y/(x^2 + y^2)$

27. $v = -e^{-2x} \sin 2y$ **28.** $u = \cos 3x \cosh 3y$

29. $u = \exp(-(x^2 - y^2)/2) \cos xy$

30. $v = \cos 2x \sinh 2y$

31–35 **Special Function Values.** Find the value of:

31. $\cos(3 - i)$ **32.** $\text{Ln}(0.6 + 0.8i)$

33. $\tan i$

34. $\sinh(1 + \pi i)$, $\sin(1 + \pi i)$

35. $\cosh(\pi + \pi i)$

SUMMARY OF CHAPTER 13
Complex Numbers and Functions. Complex Differentiation

For arithmetic operations with **complex numbers**

(1) $$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

(2) $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0 \tag{4}$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The complex **exponential function** (Sec. 13.5)

$$e^z = \exp z = e^x (\cos y + i \sin y) \tag{5}$$

reduces to e^x if $z = x$ $(y = 0)$. It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 13.6)

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{6}$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \quad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}) = -i \sin iz \tag{7}$$

etc. The functions (5)–(7) are **entire**, that is, analytic everywhere in the complex plane.

The **natural logarithm** is (Sec. 13.7)

$$\ln z = \ln|z| + i \arg z = \ln|z| + i \operatorname{Arg} z \pm 2n\pi i \tag{8}$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln z of $\ln z$; thus $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$.

General powers are defined by (Sec. 13.7)

$$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0). \tag{9}$$

CHAPTER 14

Complex Integration

Chapter 13 laid the groundwork for the study of complex analysis, covered complex numbers in the complex plane, limits, and differentiation, and introduced the most important concept of analyticity. A complex function is *analytic* in some domain if it is differentiable in that domain. Complex analysis deals with such functions and their applications. The Cauchy–Riemann equations, in Sec. 13.4, were the heart of Chapter 13 and allowed a means of checking whether a function is indeed analytic. In that section, we also saw that analytic functions satisfy Laplace's equation, the most important PDE in physics.

We now consider the next part of complex calculus, that is, we shall discuss the first approach to complex integration. It centers around the very important ***Cauchy integral theorem*** (also called the *Cauchy–Goursat theorem*) in Sec. 14.2. This theorem is important because it allows, through its implied ***Cauchy integral formula*** of Sec. 14.3, the evaluation of integrals having an analytic integrand. Furthermore, the Cauchy integral formula shows the surprising result that analytic functions have derivatives of all orders. Hence, in this respect, complex analytic functions behave much more simply than real-valued functions of real variables, which may have derivatives only up to a certain order.

Complex integration is attractive for several reasons. Some basic properties of analytic functions are difficult to prove by other methods. This includes the existence of derivatives of all orders just discussed. A main practical reason for the importance of integration in the complex plane is that such integration can evaluate certain real integrals that appear in applications and that are not accessible by real integral calculus.

Finally, complex integration is used in connection with special functions, such as gamma functions (consult [GenRef1]), the error function, and various polynomials (see [GenRef10]). These functions are applied to problems in physics.

The second approach to complex integration is integration by residues, which we shall cover in Chapter 16.

Prerequisite: Chap. 13.
Section that may be omitted in a shorter course: 14.1, 14.5.
References and Answers to Problems: App. 1 Part D, App. 2.

14.1 Line Integral in the Complex Plane

As in calculus, in complex analysis we distinguish between definite integrals and indefinite integrals or antiderivatives. Here an **indefinite integral** is a function whose derivative equals a given analytic function in a region. By inverting known differentiation formulas we may find many types of indefinite integrals.

Complex definite integrals are called (complex) **line integrals**. They are written

$$\int_C f(z)\, dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve C or a portion of it (an *arc*, but we shall say "***curve***" in either case, for simplicity). This curve C in the complex plane is called the **path of integration**. We may represent C by a parametric representation

$$z(t) = x(t) = iy(t) \qquad (a \leqq t \leqq b). \tag{1}$$

The sense of increasing t is called the **positive sense** on C, and we say that C is **oriented** by (1).

For instance, $z(t) = t + 3it$ $(0 \leqq t \leqq 2)$ gives a portion (a segment) of the line $y = 3x$. The function $z(t) = 4\cos t + 4i\sin t$ $(-\pi \leqq t \leqq \pi)$ represents the circle $|z| = 4$, and so on. More examples follow below.

We assume C to be a **smooth curve**, that is, C has a continuous and nonzero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each point. Geometrically this means that C has everywhere a continuously turning tangent, as follows directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} \qquad \text{(Fig. 339).}$$

Here we use a dot since a prime $'$ denotes the derivative with respect to z.

Definition of the Complex Line Integral

This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (1), and let $f(z)$ be a continuous function given (at least) at each point of C. We now subdivide (we "***partition***") the interval $a \leqq t \leqq b$ in (1) by points

$$t_0\,(= a), \quad t_1, \quad \cdots, \quad t_{n-1}, \quad t_n\,(= b)$$

where $t_0 < t_1 < \cdots < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0, \quad z_1, \quad \cdots, \quad z_{n-1}, \quad z_n\,(= Z) \qquad \text{(Fig. 340),}$$

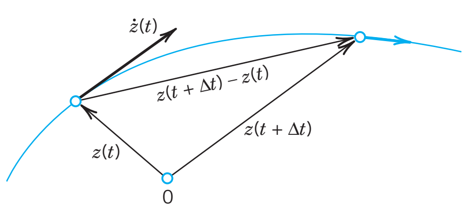


Fig. 339. Tangent vector $\dot{z}(t)$ of a curve C in the complex plane given by $z(t)$. The arrowhead on the curve indicates the *positive sense* (sense of increasing t)

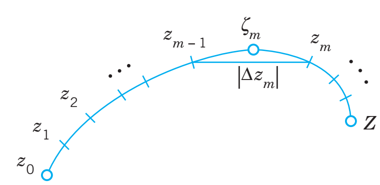


Fig. 340. Complex line integral

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ζ_1 between z_0 and z_1 (that is, $\zeta_1 = z(t)$ where t satisfies $t_0 \leqq t \leqq t_1$), a point ζ_2 between z_1 and z_2, etc. Then we form the sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\,\Delta z_m \qquad \text{where} \qquad \Delta z_m = z_m - z_{m-1}. \tag{2}$$

We do this for each $n = 2, 3, \cdots$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta z_m|$ also approaches zero. Indeed, it cannot exceed the length of the arc of C from z_{m-1} to z_m and the latter goes to zero since the arc length of the smooth curve C is a continuous function of t. The limit of the sequence of complex numbers $S_2, S_3, \cdots$ thus obtained is called the **line integral** (or simply the *integral*) of $f(z)$ over the path of integration C with the orientation given by (1). This line integral is denoted by

$$\int_C f(z)\,dz, \qquad \text{or by} \qquad \oint_C f(z)\,dz \tag{3}$$

if C is a **closed path** (one whose terminal point Z coincides with its initial point z_0, as for a circle or for a curve shaped like an 8).

General Assumption. *All paths of integration for complex line integrals are assumed to be* **piecewise smooth**, *that is, they consist of finitely many smooth curves joined end to end.*

Basic Properties Directly Implied by the Definition

1. **Linearity.** Integration is a **linear operation**, that is, we can integrate sums term by term and can take out constant factors from under the integral sign. This means that if the integrals of f_1 and f_2 over a path C exist, so does the integral of $k_1 f_1 + k_2 f_2$ over the same path and

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz. \tag{4}$$

2. **Sense reversal** in integrating over the ***same*** path, from z_0 to Z (left) and from Z to z_0 (right), introduces a minus sign as shown,

$$\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz. \tag{5}$$

3. **Partitioning of path** (see Fig. 341)

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{6}$$

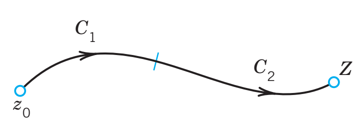


Fig. 341. Partitioning of path [formula (6)]

Existence of the Complex Line Integral

Our assumptions that $f(z)$ is continuous and C is piecewise smooth imply the existence of the line integral (3). This can be seen as follows.

As in the preceding chapter let us write $f(z) = u(x, y) + iv(x, y)$. We also set

$$\zeta_m = \xi_m + i\eta_m \qquad \text{and} \qquad \Delta z_m = \Delta x_m + i\Delta y_m.$$

Then (2) may be written

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m) \tag{7}$$

where $u = u(\zeta_m, \eta_m)$, $v = v(\zeta_m, \eta_m)$ and we sum over m from 1 to n. Performing the multiplication, we may now split up S_n into four sums:

$$S_n = \sum u\,\Delta x_m - \sum v\,\Delta y_m + i\left[\sum u\,\Delta y_m + \sum v\,\Delta x_m\right].$$

These sums are real. Since f is continuous, u and v are continuous. Hence, if we let n approach infinity in the aforementioned way, then the greatest Δx_m and Δy_m will approach zero and each sum on the right becomes a real line integral:

$$\begin{aligned} \lim_{n\to\infty} S_n &= \int_C f(z)\,dz \\ &= \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]. \end{aligned} \tag{8}$$

This shows that under our assumptions on f and C the line integral (3) exists and its value is independent of the choice of subdivisions and intermediate points ζ_m. ■

First Evaluation Method: Indefinite Integration and Substitution of Limits

This method is the analog of the evaluation of definite integrals in calculus by the well-known formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $[F'(x) = f(x)]$.

It is simpler than the next method, but it is suitable for analytic functions only. To formulate it, we need the following concept of general interest.

A domain D is called **simply connected** if every **simple closed curve** (closed curve without self-intersections) encloses only points of D.

For instance, a circular disk is simply connected, whereas an annulus (Sec. 13.3) is not simply connected. (Explain!)

THEOREM 1

Indefinite Integration of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]. \tag{9}$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

This theorem will be proved in the next section.

Simple connectedness is quite essential in Theorem 1, as we shall see in Example 5.

Since analytic functions are our main concern, and since differentiation formulas will often help in finding $F(z)$ for a given $f(z) = F'(z)$, the present method is of great practical interest.

If $f(z)$ is entire (Sec. 13.5), we can take for D the complex plane (which is certainly simply connected).

EXAMPLE 1

$$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$ ■

EXAMPLE 2

$$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097i$$ ■

EXAMPLE 3

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

since e^z is periodic with period $2\pi i$. ■

EXAMPLE 4

$\displaystyle\int_{-i}^{i} \frac{dz}{z} = \text{Ln}\, i - \text{Ln}\,(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi$. Here D is the complex plane without 0 and the negative real axis (where Ln z is not analytic). Obviously, D is a simply connected domain. ■

Second Evaluation Method: Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function.

THEOREM 2

Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right). \tag{10}$$

PROOF The left side of (10) is given by (8) in terms of real line integrals, and we show that the right side of (10) also equals (8). We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for $u[x(t), y(t)]$ and v for $v[x(t), y(t)]$. We also have $dx = \dot{x}\,dt$ *and* $dy = \dot{y}\,dt$. Consequently, in (10)

$$\begin{aligned}\int_a^b f[z(t)]\dot{z}(t)\,dt &= \int_a^b (u + iv)(\dot{x} + i\dot{y})\,dt \\ &= \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)] \\ &= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx).\end{aligned}$$ ■

COMMENT. In (7) and (8) of the existence proof of the complex line integral we referred to real line integrals. If one wants to avoid this, one can take (10) as a *definition* of the complex line integral.

Steps in Applying Theorem 2

(A) Represent the path C in the form $z(t)$ $(a \leqq t \leqq b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

EXAMPLE 5 A Basic Result: Integral of $1/z$ Around the Unit Circle

We show that by integrating $1/z$ counterclockwise around the unit circle (the circle of radius 1 and center 0; see Sec. 13.3) we obtain

(11)
$$\oint_C \frac{dz}{z} = 2\pi i \qquad (C \text{ the unit circle, counterclockwise}).$$

This is a very important result that we shall need quite often.

Solution. **(A)** We may represent the unit circle C in Fig. 330 of Sec. 13.3 by

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \leqq t \leqq 2\pi),$$

so that counterclockwise integration corresponds to an increase of t from 0 to 2π.

(B) Differentiation gives $\dot{z}(t) = ie^{it}$ (chain rule!).

(C) By substitution, $f(z(t)) = 1/z(t) = e^{-it}$.

(D) From (10) we thus obtain the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Check this result by using $z(t) = \cos t + i \sin t$.

Simple connectedness is essential in Theorem 1. Equation (9) in Theorem 1 gives 0 for any closed path because then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$. Now $1/z$ is not analytic at $z = 0$. But any *simply connected* domain containing the unit circle must contain $z = 0$, so that Theorem 1 does not apply—it is not enough that $1/z$ is analytic in an annulus, say, $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected! ■

EXAMPLE 6 Integral of $1/z^m$ with Integer Power m

Let $f(z) = (z - z_0)^m$ where m is the integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0 (Fig. 342).

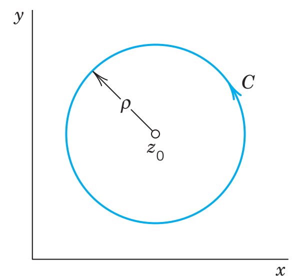


Fig. 342. Path in Example 6

Solution. We may represent C in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \qquad (0 \leqq t \leqq 2\pi).$$

Then we have

$$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it}\, dt$$

and obtain

$$\oint_C (z - z_0)^m\, dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it}\, dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt.$$

By the Euler formula (5) in Sec. 13.6 the right side equals

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\, dt + i \int_0^{2\pi} \sin(m+1)t\, dt\right].$$

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$. For integer $m \neq -1$ each of the two integrals is zero because we integrate over an interval of length 2π, equal to a period of sine and cosine. Hence the result is

$$\text{(12)} \qquad \oint_C (z - z_0)^m\, dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}). \end{cases}$$

■

Dependence on path. Now comes a very important fact. If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, ***a complex line integral depends not only on the endpoints of the path but in general also on the path itself***. The next example gives a first impression of this, and a systematic discussion follows in the next section.

EXAMPLE 7 Integral of a Nonanalytic Function. Dependence on Path

Integrate $f(z) = \operatorname{Re} z = x$ from 0 to $1 + 2i$ (a) along C^* in Fig. 343, (b) along C consisting of C_1 and C_2.

Solution. **(a)** C^* can be represented by $z(t) = t + 2it$ $(0 \leqq t \leqq 1)$. Hence $\dot{z}(t) = 1 + 2i$ and $f[z(t)] = x(t) = t$ on C^*. We now calculate

$$\int_{C^*} \operatorname{Re} z\, dz = \int_0^1 t(1 + 2i)\, dt = \frac{1}{2}(1 + 2i) = \frac{1}{2} + i.$$

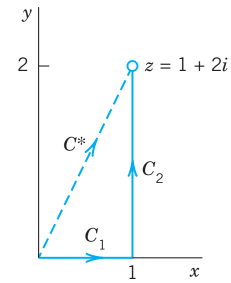


Fig. 343. Paths in Example 7

(b) We now have

$$C_1: z(t) = t, \qquad \dot{z}(t) = 1, \qquad f(z(t)) = x(t) = t \qquad (0 \leqq t \leqq 1)$$

$$C_2: z(t) = 1 + it, \qquad \dot{z}(t) = i, \qquad f(z(t)) = x(t) = 1 \qquad (0 \leqq t \leqq 2).$$

Using (6) we calculate

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} \operatorname{Re} z\, dz + \int_{C_2} \operatorname{Re} z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a). ■

Bounds for Integrals. *ML*-Inequality

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

$$\left| \int_C f(z)\, dz \right| \leqq ML \qquad \textbf{(13)} \quad (\textbf{\textit{ML}-inequality});$$

L is the length of C and M a constant such that $|f(z)| \leqq M$ everywhere on C.

PROOF Taking the absolute value in (2) and applying the generalized inequality (6*) in Sec. 13.2, we obtain

$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m \right| \leqq \sum_{m=1}^{n} |f(\zeta_m)| |\Delta z_m| \leqq M \sum_{m=1}^{n} |\Delta z_m|.$$

Now $|\Delta z_m|$ is the length of the chord whose endpoints are z_{m-1} and z_m (see Fig. 340). Hence the sum on the right represents the length L^* of the broken line of chords whose endpoints are $z_0, z_1, \cdots, z_n (= Z)$. If n approaches infinity in such a way that the greatest $|\Delta t_m|$ and thus $|\Delta z_m|$ approach zero, then L^* approaches the length L of the curve C, by the definition of the length of a curve. From this the inequality (13) follows. ■

We cannot see from (13) how close to the bound ML the actual absolute value of the integral is, but this will be no handicap in applying (13). For the time being we explain the practical use of (13) by a simple example.

EXAMPLE 8 **Estimation of an Integral**

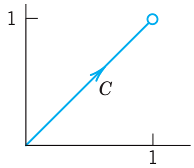


Fig. 344. Path in Example 8

Find an upper bound for the absolute value of the integral

$$\int_C z^2\, dz, \qquad C \text{ the straight-line segment from 0 to } 1+i, \text{ Fig. 344.}$$

Solution. $L = \sqrt{2}$ and $|f(z)| = |z^2| \leqq 2$ on C gives by (13)

$$\left|\int_C z^2\, dz\right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is $|-\frac{2}{3} + \frac{2}{3}i| = \frac{2}{3}\sqrt{2} = 0.9428$ (see Example 1). ■

Summary on Integration. Line integrals of $f(z)$ can always be evaluated by (10), using a representation (1) of the path of integration. If $f(z)$ is analytic, indefinite integration by (9) as in calculus will be simpler (proof in the next section).

PROBLEM SET 14.1

1–10 FIND THE PATH and sketch it.

1. $z(t) = (1 + \frac{1}{2}i)t \quad (2 \leqq t \leqq 5)$
2. $z(t) = 3 + i + (1 - i)t \quad (0 \leqq t \leqq 3)$
3. $z(t) = t + 2it^2 \quad (1 \leqq t \leqq 2)$
4. $z(t) = t + (1 - t)^2 i \quad (-1 \leqq t \leqq 1)$
5. $z(t) = 3 - i + \sqrt{10}e^{-it} \quad (0 \leqq t \leqq 2\pi)$
6. $z(t) = 1 + i + e^{-\pi i t} \quad (0 \leqq t \leqq 2)$
7. $z(t) = 2 + 4e^{\pi i t/2} \quad (0 \leqq t \leqq 2)$
8. $z(t) = 5e^{-it} \quad (0 \leqq t \leqq \pi/2)$
9. $z(t) = t + it^3 \quad (-2 \leqq t \leqq 2)$
10. $z(t) = 2\cos t + i \sin t \quad (0 \leqq t \leqq 2\pi)$

11–20 FIND A PARAMETRIC REPRESENTATION and sketch the path.

11. Segment from $(-1, 1)$ to $(1, 3)$
12. From $(0, 0)$ to $(2, 1)$ along the axes
13. Upper half of $|z - 2 + i| = 2$ from $(4, -1)$ to $(0, -1)$
14. Unit circle, clockwise
15. $x^2 - 4y^2 = 4$, the branch through $(2, 0)$
16. Ellipse $4x^2 + 9y^2 = 36$, counterclockwise
17. $|z + a + ib| = r$, clockwise
18. $y = 1/x$ from $(1, 1)$ to $(5, \frac{1}{5})$
19. Parabola $y = 1 - \frac{1}{4}x^2 \quad (-2 \leqq x \leqq 2)$
20. $4(x - 2)^2 + 5(y + 1)^2 = 20$

21–30 INTEGRATION

Integrate by the first method or state why it does not apply and use the second method. Show the details.

21. $\int_C \text{Re}\, z\, dz$, C the shortest path from $1 + i$ to $3 + 3i$
22. $\int_C \text{Re}\, z\, dz$, C the parabola $y = 1 + \frac{1}{2}(x - 1)^2$ from $1 + i$ to $3 + 3i$
23. $\int_C e^z\, dz$, C the shortest path from πi to $2\pi i$
24. $\int_C \cos 2z\, dz$, C the semicircle $|z| = \pi, x \geqq 0$ from $-\pi i$ to πi
25. $\int_C z \exp(z^2)\, dz$, C from 1 along the axes to i
26. $\int_C (z + z^{-1})\, dz$, C the unit circle, counterclockwise
27. $\int_C \sec^2 z\, dz$, any path from $\pi/4$ to $\pi i/4$
28. $\int_C \left(\frac{5}{z - 2i} - \frac{6}{(z - 2i)^2}\right) dz$, C the circle $|z - 2i| = 4$, clockwise
29. $\int_C \text{Im}\, z^2\, dz$ counterclockwise around the triangle with vertices $0, 1, i$
30. $\int_C \text{Re}\, z^2\, dz$ clockwise around the boundary of the square with vertices $0, i, 1 + i, 1$
31. **CAS PROJECT. Integration.** Write programs for the two integration methods. Apply them to problems of your choice. Could you make them into a joint program that also decides which of the two methods to use in a given case?

32. Sense reversal. Verify (5) for $f(z) = z^2$, where C is the segment from $-1 - i$ to $1 + i$.

33. Path partitioning. Verify (6) for $f(z) = 1/z$ and C_1 and C_2 the upper and lower halves of the unit circle.

34. TEAM EXPERIMENT. Integration. (a) Comparison. First write a short report comparing the essential points of the two integration methods.

(b) Comparison. Evaluate $\int_C f(z)\,dz$ by Theorem 1 and check the result by Theorem 2, where:

(i) $f(z) = z^4$ and C is the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane,

(ii) $f(z) = e^{2z}$ and C is the shortest path from 0 to $1 + 2i$.

(c) Continuous deformation of path. Experiment with a family of paths with common endpoints, say, $z(t) = t + ia \sin t$, $0 \leqq t \leqq \pi$, with real parameter a. Integrate nonanalytic functions (Re z, Re (z^2), etc.) and explore how the result depends on a. Then take analytic functions of your choice. (Show the details of your work.) Compare and comment.

(d) Continuous deformation of path. Choose another family, for example, semi-ellipses $z(t) = a \cos t + i \sin t$, $-\pi/2 \leqq t \leqq \pi/2$, and experiment as in (c).

35. *ML*-inequality. Find an upper bound of the absolute value of the integral in Prob. 21.

14.2 Cauchy's Integral Theorem

This section is the focal point of the chapter. We have just seen in Sec. 14.1 that a line integral of a function $f(z)$ generally depends not merely on the endpoints of the path, but also on the choice of the path itself. This dependence often complicates situations. Hence conditions under which this does ***not*** occur are of considerable importance. Namely, if $f(z)$ is analytic in a domain D and D is simply connected (see Sec. 14.1 and also below), then the integral will not depend on the choice of a path between given points. This result (Theorem 2) follows from Cauchy's integral theorem, along with other basic consequences that make ***Cauchy's integral theorem the most important theorem in this chapter*** and fundamental throughout complex analysis.

Let us continue our discussion of simple connectedness which we started in Sec. 14.1.

1. A **simple closed path** is a closed path (defined in Sec. 14.1) that does not intersect or touch itself as shown in Fig. 345. For example, a circle is simple, but a curve shaped like an 8 is not simple.

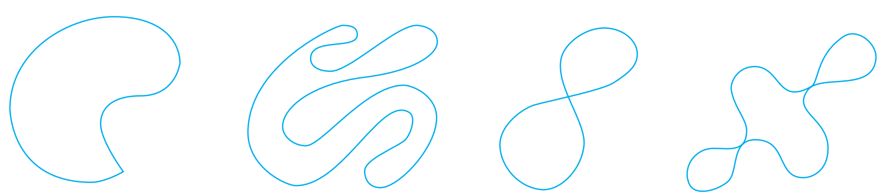


Fig. 345. Closed paths

2. A **simply connected domain** D in the complex plane is a domain (Sec. 13.3) such that every simple closed path in D encloses only points of D. *Examples:* The interior of a circle ("open disk"), ellipse, or any simple closed curve. A domain that is not simply connected is called **multiply connected.** *Examples:* An annulus (Sec. 13.3), a disk without the center, for example, $0 < |z| < 1$. See also Fig. 346.

More precisely, a **bounded domain** D (that is, a domain that lies entirely in some circle about the origin) is called ***p*-fold connected** if its boundary consists of p closed

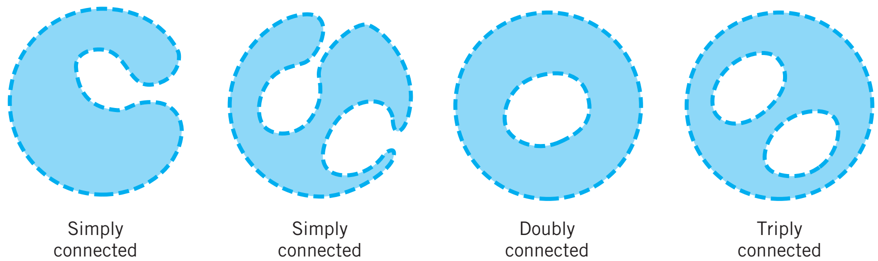


Fig. 346. Simply and multiply connected domains

connected sets without common points. These sets can be curves, segments, or single points (such as $z = 0$ for $0 < |z| < 1$, for which $p = 2$). Thus, D has $p - 1$ "**holes**," where "hole" may also mean a segment or even a single point. Hence an annulus is doubly connected ($p = 2$).

THEOREM 1

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0. \tag{1}$$

See Fig. 347.

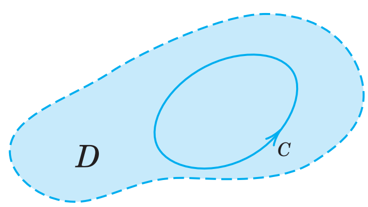


Fig. 347. Cauchy's integral theorem

Before we prove the theorem, let us consider some examples in order to really understand what is going on. A simple closed path is sometimes called a *contour* and an integral over such a path a **contour integral**. Thus, (1) and our examples involve contour integrals.

EXAMPLE 1 **Entire Functions**

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all z). ■

EXAMPLE 2 **Points Outside the Contour Where $f(x)$ is Not Analytic**

$$\oint_C \sec z\,dz = 0, \qquad \oint_C \frac{dz}{z^2 + 4} = 0$$

where C is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C; none lies on C or inside C. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2i$ outside C. ■

EXAMPLE 3 Nonanalytic Function

$$\oint_C \bar{z}\, dz = \int_0^{2\pi} e^{-it} ie^{it}\, dt = 2\pi i$$

where C: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic. ■

EXAMPLE 4 Analyticity Sufficient, Not Necessary

$$\oint_C \frac{dz}{z^2} = 0$$

where C is the unit circle. This result does *not* follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence *the condition that f be analytic in D is sufficient rather than necessary for* (1) *to be true.* ■

EXAMPLE 5 Simple Connectedness Essential

$$\oint_C \frac{dz}{z} = 2\pi i$$

for counterclockwise integration around the unit circle (see Sec. 14.1). C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $1/z$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence ***the condition that the domain D be simply connected is essential.***

In other words, by Cauchy's theorem, if $f(z)$ is analytic on a simple closed path C and everywhere inside C, with no exception, not even a single point, then (1) holds. The point that causes trouble here is $z = 0$ where $1/z$ is not analytic. ■

PROOF Cauchy proved his integral theorem under the additional assumption that the derivative $f'(z)$ is continuous (which is true, but would need an extra proof). His proof proceeds as follows. From (8) in Sec. 14.1 we have

$$\oint_C f(z)\, dz = \oint_C (u\, dx - v\, dy) + i \oint_C (u\, dy + v\, dx).$$

Since $f(z)$ is analytic in D, its derivative $f'(z)$ exists in D. Since $f'(z)$ is assumed to be continuous, (4) and (5) in Sec. 13.4 imply that u and v have *continuous* partial derivatives in D. Hence Green's theorem (Sec. 10.4) (with u and $-v$ instead of F_1 and F_2) is applicable and gives

$$\oint_C (u\, dx - v\, dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

where R is the region bounded by C. The second Cauchy–Riemann equation (Sec. 13.4) shows that the integrand on the right is identically zero. Hence the integral on the left is zero. In the same fashion it follows by the use of the first Cauchy–Riemann equation that the last integral in the above formula is zero. This completes Cauchy's proof. ■

Goursat's proof ***without the condition that*** $f'(z)$ ***is continuous***[1] is much more complicated. We leave it optional and include it in App. 4.

[1] ÉDOUARD GOURSAT (1858–1936), French mathematician who made important contributions to complex analysis and PDEs. Cauchy published the theorem in 1825. The removal of that condition by Goursat (see *Transactions Amer. Math Soc.*, vol. 1, 1900) is quite important because, for instance, derivatives of analytic functions are also analytic. Because of this, Cauchy's integral theorem is also called Cauchy–Goursat theorem.

Independence of Path

We know from the preceding section that the value of a line integral of a given function $f(z)$ from a point z_1 to a point z_2 will in general depend on the path C over which we integrate, not merely on z_1 and z_2. It is important to characterize situations in which this difficulty of path dependence does not occur. This task suggests the following concept. We call an integral of $f(z)$ **independent of path in a domain D** if for every z_1, z_2 in D its value depends (besides on $f(z)$, of course) only on the initial point z_1 and the terminal point z_2, but not on the choice of the path C in D [so that every path in D from z_1 to z_2 gives the same value of the integral of $f(z)$].

THEOREM 2

Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

PROOF Let z_1 and z_2 be any points in D. Consider two paths C_1 and C_2 in D from z_1 to z_2 without further common points, as in Fig. 348. Denote by C_2^* the path C_2 with the orientation reversed (Fig. 349). Integrate from z_1 over C_1 to z_2 and over C_2^* back to z_1. This is a simple closed path, and Cauchy's theorem applies under our assumptions of the present theorem and gives zero:

$$\int_{C_1} f\,dz + \int_{C_2^*} f\,dz = 0, \qquad \text{thus} \qquad \int_{C_1} f\,dz = -\int_{C_2^*} f\,dz. \tag{2'}$$

But the minus sign on the right disappears if we integrate in the reverse direction, from z_1 to z_2, which shows that the integrals of $f(z)$ over C_1 and C_2 are equal,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad \text{(Fig. 348).} \tag{2}$$

This proves the theorem for paths that have only the endpoints in common. For paths that have finitely many further common points, apply the present argument to each "loop" (portions of C_1 and C_2 between consecutive common points; four loops in Fig. 350). For paths with infinitely many common points we would need additional argumentation not to be presented here.

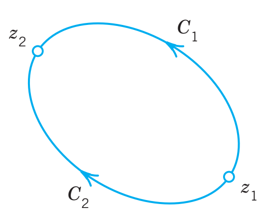


Fig. 348. Formula (2)

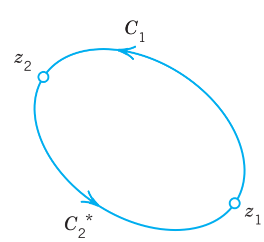


Fig. 349. Formula (2′)

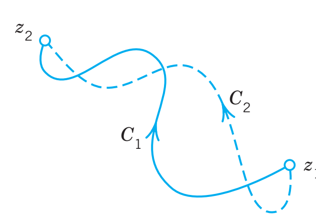


Fig. 350. Paths with more common points

Principle of Deformation of Path

This idea is related to path independence. We may imagine that the path C_2 in (2) was obtained from C_1 by continuously moving C_1 (with ends fixed!) until it coincides with C_2. Figure 351 shows two of the infinitely many intermediate paths for which the integral always retains its value (because of Theorem 2). Hence we may impose a continuous deformation of the path of an integral, keeping the ends fixed. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the **principle of deformation of path**.

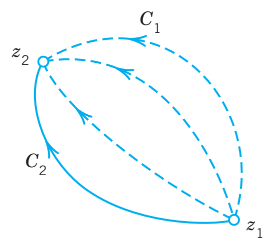


Fig. 351. Continuous deformation of path

EXAMPLE 6 A Basic Result: Integral of Integer Powers

From Example 6 in Sec. 14.1 and the principle of deformation of path it follows that

$$\oint (z - z_0)^m \, dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases} \tag{3}$$

for counterclockwise integration around ***any simple closed path containing z_0 in its interior***.

Indeed, the circle $|z - z_0| = \rho$ in Example 6 of Sec. 14.1 can be continuously deformed in two steps into a path as just indicated, namely, by first deforming, say, one semicircle and then the other one. (Make a sketch). ■

Existence of Indefinite Integral

We shall now justify our indefinite integration method in the preceding section [formula (9) in Sec. 14.1]. The proof will need Cauchy's integral theorem.

THEOREM 3

Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D—thus, $F'(z) = f(z)$—which is analytic in D, and for all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by formula (9) *in* Sec. 14.1.

PROOF The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from any z_0 in D to any z in D is independent of path in D. We keep z_0 fixed. Then this integral becomes a function of z, call if $F(z)$,

$$F(z) = \int_{z_0}^{z} f(z^*) \, dz^* \tag{4}$$

which is uniquely determined. We show that this $F(z)$ is analytic in D and $F'(z) = f(z)$. The idea of doing this is as follows. Using (4) we form the difference quotient

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)\, dz^* - \int_{z_0}^{z} f(z^*)\, dz^*\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)\, dz^*. \tag{5}$$

We now subtract $f(z)$ from (5) and show that the resulting expression approaches zero as $\Delta z \to 0$. The details are as follows.

We keep z fixed. Then we choose $z + \Delta z$ in D so that the whole segment with endpoints z and $z + \Delta z$ is in D (Fig. 352). This can be done because D is a domain, hence it contains a neighborhood of z. We use this segment as the path of integration in (5). Now we subtract $f(z)$. This is a constant because z is kept fixed. Hence we can write

$$\int_{z}^{z+\Delta z} f(z)\, dz^* = f(z)\int_{z}^{z+\Delta z} dz^* = f(z)\,\Delta z. \qquad \text{Thus} \qquad f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, dz^*.$$

By this trick and from (5) we get a single integral:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(z^*) - f(z)]\, dz^*.$$

Since $f(z)$ is analytic, it is continuous (see Team Project (24d) in Sec. 13.3). An $\epsilon > 0$ being given, we can thus find a $\delta > 0$ such that $|f(z^*) - f(z)| < \epsilon$ when $|z^* - z| < \delta$. Hence, letting $|\Delta z| < \delta$, we see that the *ML*-inequality (Sec. 14.1) yields

$$\left|\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z} [f(z^*) - f(z)]\, dz^*\right| \leqq \frac{1}{|\Delta z|}\,\epsilon|\Delta z| = \epsilon.$$

By the definition of limit and derivative, this proves that

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

Since z is any point in D, this implies that $F(z)$ is analytic in D and is an indefinite integral or antiderivative of $f(z)$ in D, written

$$F(z) = \int f(z)\, dz.$$

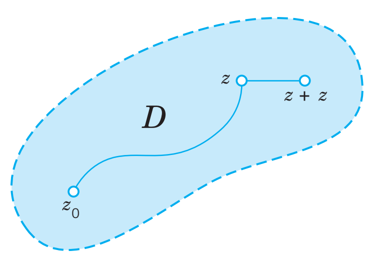


Fig. 352. Path of integration

Also, if $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in D; hence $F(z) - G(z)$ is constant in D (see Team Project 30 in Problem Set 13.4). That is, two indefinite integrals of $f(z)$ can differ only by a constant. The latter drops out in (9) of Sec. 14.1, so that we can use any indefinite integral of $f(z)$. This proves Theorem 3. ■

Cauchy's Integral Theorem for Multiply Connected Domains

Cauchy's theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** D with outer boundary curve C_1 and inner C_2 (Fig. 353). If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz \qquad (6) \quad \text{(Fig. 353)}$$

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of C_2 belongs to D^*).

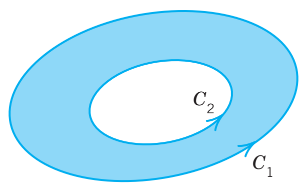


Fig. 353. Paths in (5)

PROOF By two cuts $\widetilde{C}_1$ and $\widetilde{C}_2$ (Fig. 354) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic. By Cauchy's integral theorem the integral over the entire boundary of D_1 (taken in the sense of the arrows in Fig. 354) is zero, and so is the integral over the boundary of D_2, and thus their sum. In this sum the integrals over the cuts $\widetilde{C}_1$ and $\widetilde{C}_2$ cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over C_1 (counterclockwise) and C_2 (clockwise; see Fig. 354); hence by reversing the integration over C_2 (to counterclockwise) we have

$$\oint_{C_1} f\, dz - \oint_{C_2} f\, dz = 0$$

and (6) follows. ■

For domains of higher connectivity the idea remains the same. Thus, for a **triply connected domain** we use three cuts $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$ (Fig. 355). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over C_1 (counterclockwise) and C_2, C_3 (clockwise) is zero. Hence the integral over C_1 equals the sum of the integrals over C_2 and C_3, all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

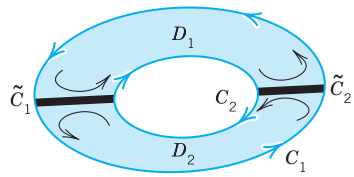


Fig. 354. Doubly connected domain

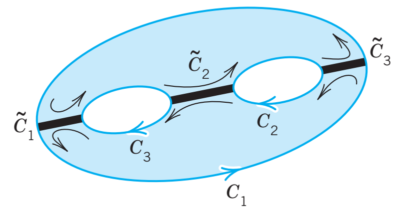


Fig. 355. Triply connected domain

PROBLEM SET 14.2

1–8 COMMENTS ON TEXT AND EXAMPLES

1. Cauchy's Integral Theorem. Verify Theorem 1 for the integral of z^2 over the boundary of the square with vertices $\pm 1 \pm i$. *Hint.* Use deformation.

2. For what contours C will it follow from Theorem 1 that

$$\textbf{(a)} \quad \int_C \frac{dz}{z} = 0, \qquad \textbf{(b)} \quad \int_C \frac{\exp(1/z^2)}{z^2 + 16}\, dz = 0\,?$$

3. Deformation principle. Can we conclude from Example 4 that the integral is also zero over the contour in Prob. 1?

4. If the integral of a function over the unit circle equals 2 and over the circle of radius 3 equals 6, can the function be analytic everywhere in the annulus $1 < |z| < 3$?

5. Connectedness. What is the connectedness of the domain in which $(\cos z^2)/(z^4 + 1)$ is analytic?

6. Path independence. Verify Theorem 2 for the integral of e^z from 0 to $1 + i$ **(a)** over the shortest path and **(b)** over the x-axis to 1 and then straight up to $1 + i$.

7. Deformation. Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ over **(a)** $|z - 2| = 2$ and **(b)** $|z - 2| = 3$ is zero?

8. TEAM EXPERIMENT. Cauchy's Integral Theorem.

(a) Main Aspects. Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.

(b) Partial fractions. Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where

$$\text{(i)} \quad f(z) = \frac{2z + 3i}{z^2 + \frac{1}{4}} \qquad \text{(ii)} \quad f(z) = \frac{z + 1}{z^2 + 2z}\,.$$

(c) Deformation of path. Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \leqq t \leqq 1$, a a real constant, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., z, $\text{Im}\, z$, z^2, $\text{Re}\, z^2$, $\text{Im}\, z^2$, etc.).

9–19 CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

9. $f(z) = \exp(-z^2)$

10. $f(z) = \tan \frac{1}{4}z$

11. $f(z) = 1/(2z - 1)$

12. $f(z) = \bar{z}^3$

13. $f(z) = 1/(z^4 - 1.1)$

14. $f(z) = 1/\bar{z}$

15. $f(z) = \text{Im}\, z$

16. $f(z) = 1/(\pi z - 1)$

17. $f(z) = 1/|z|^2$

18. $f(z) = 1/(4z - 3)$

19. $f(z) = z^3 \cot z$

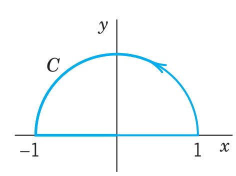
20–30 FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

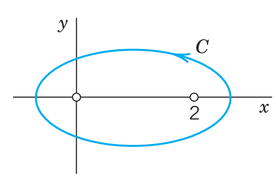
20. $\oint_C \text{Ln}\,(1 - z)\, dz$, C the boundary of the parallelogram with vertices $\pm i$, $\pm(1 + i)$.

21. $\oint_C \dfrac{dz}{z - 3i}$, C the circle $|z| = \pi$ counterclockwise.

22. $\oint_C \text{Re}\, z\, dz$, $\quad C$:

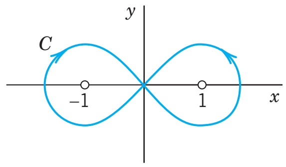


23. $\oint_C \dfrac{2z - 1}{z^2 - z}\, dz$, $\quad C$:



Use partial fractions.

24. $\oint_C \frac{dz}{z^2 - 1}$, C:



Use partial fractions.

25. $\oint_C \frac{e^z}{z}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

26. $\oint_C \coth \frac{1}{2}z\,dz$, C the circle $|z - \frac{1}{2}\pi i| = 1$ clockwise.

27. $\oint_C \frac{\cos z}{z}\,dz$, C consists of $|z| = 1$ counterclockwise and $|z| = 3$ clockwise.

28. $\oint_C \frac{\tan \frac{1}{2}z}{z^4 - 16}\,dz$, C the boundary of the square with vertices $\pm 1, \pm i$ clockwise.

29. $\oint_C \frac{\sin z}{z + 2iz}\,dz$, C: $|z - 4 - 2i| = 5.5$ clockwise.

30. $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\,dz$, C: $|z - 2| = 4$ clockwise. Use partial fractions.

14.3 Cauchy's Integral Formula

Cauchy's integral theorem leads to Cauchy's integral formula. This formula is useful for evaluating integrals as shown in this section. It has other important roles, such as in proving the surprising fact that analytic functions have derivatives of all orders, as shown in the next section, and in showing that all analytic functions have a Taylor series representation (to be seen in Sec. 15.4).

THEOREM 1

Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0) \qquad (1) \quad \textbf{(Cauchy's integral formula)}$$

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz \qquad (1^*) \quad \textbf{(Cauchy's integral formula).}$$

PROOF By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

$$\oint_C \frac{f(z)}{z - z_0}\,dz = f(z_0)\oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\,dz. \qquad (2)$$

The first term on the right equals $f(z_0) \cdot 2\pi i$, which follows from Example 6 in Sec. 14.2 with $m = -1$. If we can show that the second integral on the right is zero, then it would prove the theorem. Indeed, we can. The integrand of the second integral is analytic, except at z_0. Hence, by (6) in Sec. 14.2, we can replace C by a small circle K of radius ρ and center z_0 (Fig. 357), without altering the value of the integral. Since $f(z)$ is analytic, it is continuous (Team Project 24, Sec. 13.3). Hence, an $\epsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z in the disk $|z - z_0| < \delta$. Choosing the radius ρ of K smaller than δ, we thus have the inequality

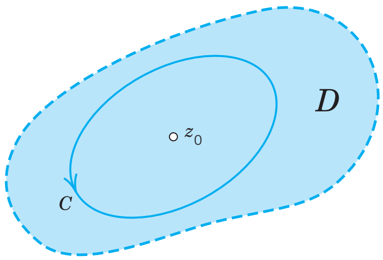


Fig. 356. Cauchy's integral formula

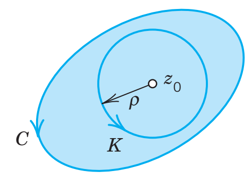


Fig. 357. Proof of Cauchy's integral formula

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\epsilon}{\rho}$$

at each point of K. The length of K is $2\pi\rho$. Hence, by the ML-inequality in Sec. 14.1,

$$\left|\oint_K \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < \frac{\epsilon}{\rho}\,2\pi\rho = 2\pi\epsilon.$$

Since ϵ (> 0) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved. ■

EXAMPLE 1 Cauchy's Integral Formula

$$\oint_C \frac{e^z}{z - 2}\,dz = 2\pi i e^z\Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem). ■

EXAMPLE 2 Cauchy's Integral Formula

$$\oint_C \frac{z^3 - 6}{2z - i}\,dz = \oint_C \frac{\frac{1}{2}z^3 - 3}{z - \frac{1}{2}i}\,dz$$

$$= 2\pi i\left[\tfrac{1}{2}z^3 - 3\right]\Big|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside } C). \quad ■$$

EXAMPLE 3 Integration Around Different Contours

Integrate

$$g(z) = \frac{z^2 + 1}{z^2 - 1} = \frac{z^2 + 1}{(z + 1)(z - 1)}$$

counterclockwise around each of the four circles in Fig. 358.

Solution. $g(z)$ is not analytic at -1 and 1. These are the points we have to watch for. We consider each circle separately.

(a) The circle $|z-1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence in (1) we have to write

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1};$$

thus

$$f(z) = \frac{z^2+1}{z+1}$$

and (1) gives

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

(b) gives the same as (a) by the principle of deformation of path.

(c) The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1). This gives a factor $z - z_0 = z + 1$ in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z-1}\,\frac{1}{z+1};$$

thus

$$f(z) = \frac{z^2+1}{z-1}.$$

Compare this for a minute with the previous expression and then go on:

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i \left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

(d) gives 0. Why? ■

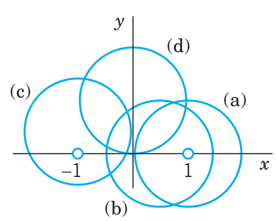


Fig. 358. Example 3

Multiply connected domains can be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 (Fig. 359) and z_0 is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz, \tag{3}$$

where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 359.

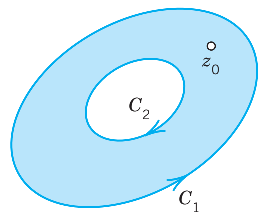


Fig. 359. Formula (3)

PROBLEM SET 14.3

1–4 CONTOUR INTEGRATION

Integrate $z^2/(z^2 - 1)$ by Cauchy's formula counterclockwise around the circle.

1. $|z + 1| = 1$

2. $|z - 1 - i| = \pi/2$

3. $|z + i| = 1.4$

4. $|z + 5 - 5i| = 7$

5–8 Integrate the given function around the unit circle.

5. $(\cos 3z)/(6z)$

6. $e^{2z}/(\pi z - i)$

7. $z^3/(2z - i)$

8. $(z^2 \sin z)/(4z - 1)$

9. CAS EXPERIMENT. Experiment to find out to what extent your CAS can do contour integration. For this, use **(a)** the second method in Sec. 14.1 and **(b)** Cauchy's integral formula.

10. TEAM PROJECT. Cauchy's Integral Theorem. Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing z_0 (as in Fig. 356) and taking the limit as in the text. Choose

$$\textbf{(a)} \quad \oint_C \frac{z^3 - 6}{z - \frac{1}{2}i}\, dz, \qquad \textbf{(b)} \quad \oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\, dz,$$

and **(c)** another example of your choice.

11–19 FURTHER CONTOUR INTEGRALS

Integrate counterclockwise or as indicated. Show the details.

11. $\displaystyle\oint_C \frac{dz}{z^2 + 4}$, $\quad C: 4x^2 + (y - 2)^2 = 4$

12. $\displaystyle\oint_C \frac{z}{z^2 + 4z + 3}\, dz$, $\quad C$ the circle with center -1 and radius 2

13. $\displaystyle\oint_C \frac{z + 2}{z - 2}\, dz$, $\quad C: |z - 1| = 2$

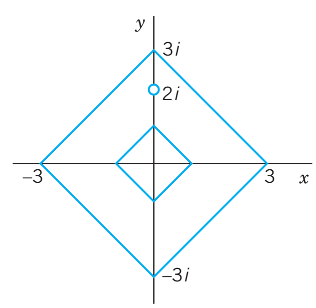
14. $\displaystyle\oint_C \frac{e^z}{ze^z - 2iz}\, dz$, $\quad C: |z| = 0.6$

15. $\displaystyle\oint_C \frac{\cosh (z^2 - \pi i)}{z - \pi i}\, dz$, $\quad C$ the boundary of the square with vertices ± 2, ± 2, $\pm 4i$.

16. $\displaystyle\oint_C \frac{\tan z}{z - i}\, dz$, $\quad C$ the boundary of the triangle with vertices 0 and $\pm 1 + 2i$.

17. $\displaystyle\oint_C \frac{\operatorname{Ln}(z + 1)}{z^2 + 1}\, dz$, $\quad C: |z - i| = 1.4$

18. $\displaystyle\oint_C \frac{\sin z}{4z^2 - 8iz}\, dz$, $\quad C$ consists of the boundaries of the squares with vertices ± 3, $\pm 3i$ counterclockwise and ± 1, $\pm i$ clockwise (see figure).



Problem 18

19. $\displaystyle\oint_C \frac{\exp z^2}{z^2(z - 1 - i)}\, dz$, $\quad C$ consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

20. Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1}\, dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

14.4 Derivatives of Analytic Functions

As mentioned, a surprising fact is that complex analytic functions have derivatives of all orders. This differs completely from real calculus. Even if a real function is once differentiable we cannot conclude that it is twice differentiable nor that any of its higher derivatives exist. This makes the behavior of complex analytic functions simpler than real functions in this aspect. To prove the surprising fact we use Cauchy's integral formula.

THEOREM 1

Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$(1') \qquad f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

$$(1'') \qquad f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3}\, dz$$

and in general

$$\textbf{(1)} \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C (Fig. 360).

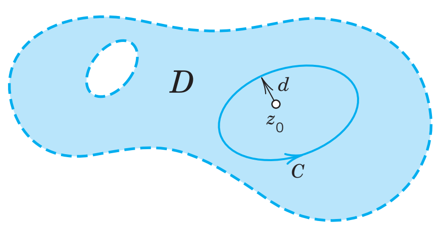


Fig. 360. Theorem 1 and its proof

COMMENT. For memorizing (1), it is useful to observe that these formulas are obtained formally by differentiating the Cauchy formula (1*), Sec. 14.3, under the integral sign *with respect to z_0*.

PROOF We prove (1′), starting from the definition of the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

On the right we represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)}\, dz - \oint_C \frac{f(z)}{z - z_0}\, dz\right].$$

We now write the two integrals as a single integral. Taking the common denominator gives the numerator $f(z)\{z - z_0 - [z - (z_0 + \Delta z)]\} = f(z)\,\Delta z$, so that a factor Δz drops out and we get

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz.$$

Clearly, we can now establish (1′) by showing that, as $\Delta z \to 0$, the integral on the right approaches the integral in (1′). To do this, we consider the difference between these two integrals. We can write this difference as a single integral by taking the common denominator and simplifying the numerator (as just before). This gives

$$\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz - \oint_C \frac{f(z)}{(z - z_0)^2}\, dz = \oint_C \frac{f(z)\,\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz.$$

We show by the *ML*-inequality (Sec. 14.1) that the integral on the right approaches zero as $\Delta z \to 0$.

Being analytic, the function $f(z)$ is continuous on C, hence bounded in absolute value, say, $|f(z)| \leqq K$. Let d be the smallest distance from z_0 to the points of C (see Fig. 360). Then for all z on C,

$$|z - z_0|^2 \geqq d^2, \qquad \text{hence} \qquad \frac{1}{|z - z_0|^2} \leqq \frac{1}{d^2}.$$

Furthermore, by the triangle inequality for all z on C we then also have

$$d \leqq |z - z_0| = |z - z_0 - \Delta z + \Delta z| \leqq |z - z_0 - \Delta z| + |\Delta z|.$$

We now subtract $|\Delta z|$ on both sides and let $|\Delta z| \leqq d/2$, so that $-|\Delta z| \geqq -d/2$. Then

$$\tfrac{1}{2}d \leqq d - |\Delta z| \leqq |z - z_0 - \Delta z|. \qquad \text{Hence} \qquad \frac{1}{|z - z_0 - \Delta z|} \leqq \frac{2}{d}.$$

Let L be the length of C. If $|\Delta z| \leqq d/2$, then by the *ML*-inequality

$$\left|\oint_C \frac{f(z)\,\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz\right| \leqq KL\,|\Delta z|\,\frac{2}{d}\cdot\frac{1}{d^2}.$$

This approaches zero as $\Delta z \to 0$. Formula (1′) is proved.

Note that we used Cauchy's integral formula (1*), Sec. 14.3, but if all we had known about $f(z_0)$ is the fact that it can be represented by (1*), Sec. 14.3, our argument would have established the existence of the derivative $f'(z_0)$ of $f(z)$. This is essential to the continuation and completion of this proof, because it implies that $(1'')$ can be proved by a similar argument, with f replaced by f', and that the general formula (1) follows by induction. ■

Applications of Theorem 1

EXAMPLE 1 Evaluation of Line Integrals

From $(1')$, for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi. \qquad ■$$

EXAMPLE 2 From $(1'')$, for any contour enclosing the point $-i$ we obtain by counterclockwise integration

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \pi i(z^4 - 3z^2 + 6)''\Big|_{z=-i} = \pi i[12z^2 - 6]_{z=-i} = -18\pi i. \qquad ■$$

EXAMPLE 3 By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1}$$

$$= 2\pi i\,\frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Big|_{z=1} = \frac{6e\pi}{25}\,i \approx 2.050i. \qquad ■$$

Cauchy's Inequality. Liouville's and Morera's Theorems

We develop other general results about analytic functions, further showing the versatility of Cauchy's integral theorem.

Cauchy's Inequality. Theorem 1 yields a basic inequality that has many applications. To get it, all we have to do is to choose for C in (1) a circle of radius r and center z_0 and apply the ML-inequality (Sec. 14.1); with $|f(z)| \leqq M$ on C we obtain from (1)

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi}\,M\,\frac{1}{r^{n+1}}\,2\pi r.$$

This gives **Cauchy's inequality**

$$|f^{(n)}(z_0)| \leqq \frac{n!M}{r^n}. \qquad (2)$$

To gain a first impression of the importance of this inequality, let us prove a famous theorem on entire functions (definition in Sec. 13.5). (For Liouville, see Sec. 11.5.)

THEOREM 2 Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

PROOF By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using (2), we see that $|f'(z_0)| < K/r$. Since $f(z)$ is entire, this holds for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = u_x + iv_x = 0$ for all z (see (4) in Sec. 13.4), hence $u_x = v_x = 0$, and $u_y = v_y = 0$ by the Cauchy–Riemann equations. Thus $u = \text{const}$, $v = \text{const}$, and $f = u + iv = \text{const}$ for all z. This completes the proof. ■

Another very interesting consequence of Theorem 1 is

THEOREM 3

Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz = 0 \tag{3}$$

for every closed path in D, then $f(z)$ is analytic in D.

PROOF In Sec. 14.2 we showed that if $f(z)$ is analytic in a simply connected domain D, then

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in D is zero; from these assumptions we concluded that $F(z)$ is analytic. By Theorem 1, the derivative of $F(z)$ is analytic, that is, $f(z)$ is analytic in D, and Morera's theorem is proved. ■

This completes Chapter 14.

PROBLEM SET 14.4

1–7 CONTOUR INTEGRATION. UNIT CIRCLE

Integrate counterclockwise around the unit circle.

1. $\oint_C \frac{\sin z}{z^4}\,dz$

2. $\oint_C \frac{z^6}{(2z-1)^6}\,dz$

3. $\oint_C \frac{e^z}{z^n}\,dz, \quad n = 1, 2, \cdots$

4. $\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\,dz$

5. $\oint_C \frac{\cosh 2z}{(z-\frac{1}{2})^4}\,dz$

6. $\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2}$

7. $\oint_C \frac{\cos z}{z^{2n+1}}\,dz, \quad n = 0, 1, \cdots$

8–19 INTEGRATION. DIFFERENT CONTOURS

Integrate. Show the details. *Hint.* Begin by sketching the contour. Why?

8. $\oint_C \frac{z^3 + \sin z}{(z-i)^3}\,dz$, C the boundary of the square with vertices $\pm 2, \pm 2i$ counterclockwise.

9. $\oint_C \frac{\tan \pi z}{z^2}\,dz$, C the ellipse $16x^2 + y^2 = 1$ clockwise.

10. $\oint_C \frac{4z^3 - 6}{z(z-1-i)^2}\,dz$, C consists of $|z| = 3$ counterclockwise and $|z| = 1$ clockwise.

---

[2] GIACINTO MORERA (1856–1909), Italian mathematician who worked in Genoa and Turin.

11. $\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz$, C: $|z-i| = 2$ counterclockwise.

12. $\oint_C \frac{\exp(z^2)}{z(z-2i)^2}\,dz$, C: $z - 3i| = 2$ clockwise.

13. $\oint_C \frac{\operatorname{Ln} z}{(z-2)^2}\,dz$, C: $|z-3| = 2$ counterclockwise.

14. $\oint_C \frac{\operatorname{Ln}(z+3)}{(z-2)(z+1)^2}\,dz$, C the boundary of the square with vertices ± 1.5, $\pm 1.5i$, counterclockwise.

15. $\oint_C \frac{\cosh 4z}{(z-4)^3}\,dz$, C consists of $|z| = 6$ counterclockwise and $|z-3| = 2$ clockwise.

16. $\oint_C \frac{e^{4z}}{z(z-2i)^2}\,dz$, C consists of $|z-i| = 3$ counterclockwise and $|z| = 1$ clockwise.

17. $\oint_C \frac{e^{-z}\sin z}{(z-4)^3}\,dz$, C consists of $|z| = 5$ counterclockwise and $|z-3| = \frac{3}{2}$ clockwise.

18. $\oint_C \frac{\sinh z}{z^n}\,dz$, C: $|z| = 1$ counterclockwise, n integer.

19. $\oint_C \frac{e^{3z}}{(4z-\pi i)^3}\,dz$, C: $|z| = 1$, counterclockwise.

20. **TEAM PROJECT. Theory on Growth**

(a) **Growth of entire functions.** If $f(z)$ is not a constant and is analytic for all (finite) z, and R and M are any positive real numbers (no matter how large), show that there exist values of z for which $|z| > R$ and $|f(z)| > M$. *Hint.* Use Liouville's theorem.

(b) **Growth of polynomials.** If $f(z)$ is a polynomial of degree $n > 0$ and M is an arbitrary positive real number (no matter how large), show that there exists a positive real number R such that $|f(z)| > M$ for all $|z| > R$.

(c) **Exponential function.** Show that $f(z) = e^x$ has the property characterized in (a) but does not have that characterized in (b).

(d) **Fundamental theorem of algebra.** *If $f(z)$ is a polynomial in z, not a constant, then $f(z) = 0$ for at least one value of z.* Prove this. *Hint.* Use (a).

CHAPTER 14 REVIEW QUESTIONS AND PROBLEMS

1. What is a parametric representation of a curve? What is its advantage?
2. What did we assume about paths of integration $z = z(t)$? What is $\dot{z} = dz/dt$ geometrically?
3. State the definition of a complex line integral from memory.
4. Can you remember the relationship between complex and real line integrals discussed in this chapter?
5. How can you evaluate a line integral of an analytic function? Of an arbitrary continous complex function?
6. What value do you get by counterclockwise integration of $1/z$ around the unit circle? You should remember this. It is basic.
7. Which theorem in this chapter do you regard as most important? State it precisely from memory.
8. What is independence of path? Its importance? State a basic theorem on independence of path in complex.
9. What is deformation of path? Give a typical example.
10. Don't confuse Cauchy's integral theorem (also known as ***Cauchy–Goursat theorem***) and Cauchy's integral formula. State both. How are they related?
11. What is a doubly connected domain? How can you extend Cauchy's integral theorem to it?
12. What do you know about derivatives of analytic functions?
13. How did we use integral formulas for derivatives in evaluating integrals?
14. How does the situation for analytic functions differ with respect to derivatives from that in calculus?
15. What is Liouville's theorem? To what complex functions does it apply?
16. What is Morera's theorem?
17. If the integrals of a function $f(z)$ over each of the two boundary circles of an annulus D taken in the same sense have different values, can $f(z)$ be analytic everywhere in D? Give reason.
18. Is $\operatorname{Im} \oint_C f(z)\,dz = \oint_C \operatorname{Im} f(z)\,dz$? Give reason.
19. Is $\left|\oint_C f(z)\,dz\right| = \oint_C |f(z)|\,dz$?
20. How would you find a bound for the left side in Prob. 19?

21–30 INTEGRATION

Integrate by a suitable method.

21. $\int_C z \sinh(z^2)\,dz$ from 0 to $\pi i/2$.

22. $\int_C (|z| + z)\, dz$ clockwise around the unit circle.

23. $\int_C z^{-5} e^z\, dz$ counterclockwise around $|z| = \pi$.

24. $\int_C \operatorname{Re} z\, dz$ from 0 to $3 + 27i$ along $y = x^3$.

25. $\int_C \frac{\tan \pi z}{(z-1)^2}\, dz$ clockwise around $|z - 1| = 0.1$.

26. $\int_C (z^2 + \bar{z}^2)\, dz$ from $z = 0$ horizontally to $z = 2$, then vertically upward to $2 + 2i$.

27. $\int_C (z^2 + \bar{z}^2)\, dz$ from 0 to $2 + 2i$, shortest path.

28. $\oint_C \frac{\operatorname{Ln} z}{(z - 2i)^2}\, dz$ counterclockwise around $|z - 1| = \frac{1}{2}$.

29. $\oint_C \left(\frac{2}{z + 2i} + \frac{1}{z + 4i}\right) dz$ clockwise around $|z - 1| = 2.5$.

30. $\int_C \sin z\, dz$ from 0 to $(1 + i)$.

SUMMARY OF CHAPTER 14
Complex Integration

The **complex line integral** of a function $f(z)$ taken over a path C is denoted by

$$(1) \qquad \int_C f(z)\, dz \qquad \text{or, if } C \text{ is closed, also by} \qquad \oint_C f(z) \qquad \text{(Sec. 14.1).}$$

If $f(z)$ is analytic in a simply connected domain D, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

$$(2) \qquad \int_C f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

for every path C in D from a point z_0 to a point z_1 (see Sec. 14.1). These assumptions imply **independence of path**, that is, (2) depends only on z_0 and z_1 (and on $f(z)$, of course) but not on the choice of C (Sec. 14.2). The existence of an $F(z)$ such that $F'(z) = f(z)$ is proved in Sec. 14.2 by Cauchy's integral theorem (see below).

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of C, where $a \leqq t \leqq b$,

$$(3) \qquad \int_C f(z)\, dz = \int_a^b f(z(t))\dot{z}(t)\, dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Cauchy's integral theorem is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D (Sec. 14.2),

$$(4) \qquad \oint_C f(z)\, dz = 0.$$

Under the same assumptions and for any z_0 in D and closed path C in D containing z_0 in its interior we also have **Cauchy's integral formula**

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{5}$$

Furthermore, under these assumptions $f(z)$ has derivatives of all orders in D that are themselves analytic functions in D and (Sec. 14.4)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 1, 2, \cdots). \tag{6}$$

This implies *Morera's theorem* (the converse of Cauchy's integral theorem) and *Cauchy's inequality* (Sec. 14.4), which in turn implies *Liouville's theorem* that an entire function that is bounded in the whole complex plane must be constant.

CHAPTER 15

Power Series, Taylor Series

In Chapter 14, we evaluated complex integrals directly by using Cauchy's integral formula, which was derived from the famous Cauchy integral theorem. We now shift from the approach of Cauchy and Goursat to another approach of evaluating complex integrals, that is, evaluating them by residue integration. This approach, discussed in Chapter 16, first requires a thorough understanding of power series and, in particular, Taylor series. (To develop the theory of residue integration, we still use Cauchy's integral theorem!)

In this chapter, we focus on complex power series and in particular Taylor series. They are analogs of real power series and Taylor series in calculus. Section 15.1 discusses convergence tests for complex series, which are quite similar to those for real series. Thus, if you are familiar with convergence tests from calculus, you may use Sec. 15.1 as a reference section. The main results of this chapter are that complex power series represent analytic functions, as shown in Sec. 15.3, and that, conversely, every analytic function can be represented by power series, called a Taylor series, as shown in Sec. 15.4. The last section (15.5) on uniform convergence is *optional.*

Prerequisite: Chaps. 13, 14.
Sections that may be omitted in a shorter course: 15.1, 15.5.
References and Answers to Problems: App. 1 Part D, App. 2.

15.1 Sequences, Series, Convergence Tests

The basic concepts for ***complex*** sequences and series and tests for convergence and divergence are very similar to those concepts in (real) calculus. ***Thus if you feel at home with real sequences and series and want to take for granted that the ratio test also holds in complex, skip this section and go to Section 15.2.***

Sequences

The basic definitions are as in calculus. An *infinite sequence* or, briefly, a **sequence**, is obtained by assigning to each positive integer n a number z_n, called a **term** of the sequence, and is written

$$z_1, z_2, \cdots \qquad \text{or} \qquad \{z_1, z_2, \cdots\} \qquad \text{or briefly} \qquad \{z_n\}.$$

We may also write $z_0, z_1, \cdots$ or $z_2, z_3, \cdots$ or start with some other integer if convenient.

A **real sequence** is one whose terms are real.

Convergence. A **convergent sequence** $z_1, z_2, \cdots$ is one that has a limit c, written

$$\lim_{n \to \infty} z_n = c \qquad \text{or simply} \qquad z_n \to c.$$

By definition of **limit** this means that for every $\epsilon > 0$ we can find an N such that

$$|z_n - c| < \epsilon \qquad \text{for all } n > N; \tag{1}$$

geometrically, all terms z_n with $n > N$ lie in the open disk of radius ϵ and center c (Fig. 361) and only finitely many terms do not lie in that disk. [For a *real* sequence, (1) gives an open interval of length 2ϵ and real midpoint c on the real line as shown in Fig. 362.]

A **divergent sequence** is one that does not converge.

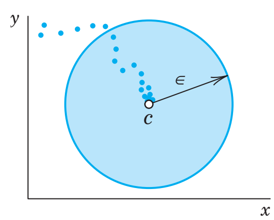


Fig. 361. Convergent complex sequence

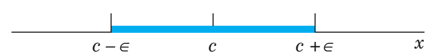


Fig. 362. Convergent real sequence

EXAMPLE 1 Convergent and Divergent Sequences

The sequence $\{i^n/n\} = \{i, -\frac{1}{2}, -i/3, \frac{1}{4}, \cdots\}$ is convergent with limit 0.

The sequence $\{i^n\} = \{i, -1, -i, 1, \cdots\}$ is divergent, and so is $\{z_n\}$ with $z_n = (1 + i)^n$. ■

EXAMPLE 2 Sequences of the Real and the Imaginary Parts

The sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - 1/n^2 + i(2 + 4/n)$ is $6i, \frac{3}{4} + 4i, \frac{8}{9} + 10i/3, \frac{15}{16} + 3i, \cdots$. (Sketch it.) It converges with the limit $c = 1 + 2i$. Observe that $\{x_n\}$ has the limit $1 = \text{Re}\, c$ and $\{y_n\}$ has the limit $2 = \text{Im}\, c$. This is typical. It illustrates the following theorem by which the convergence of a *complex* sequence can be referred back to that of the two *real* sequences of the real parts and the imaginary parts. ■

THEOREM 1 Sequences of the Real and the Imaginary Parts

A sequence $z_1, z_2, \cdots, z_n, \cdots$ of complex numbers $z_n = x_n + iy_n$ (where $n = 1, 2, \cdots$) converges to $c = a + ib$ if and only if the sequence of the real parts $x_1, x_2, \cdots$ converges to a and the sequence of the imaginary parts $y_1, y_2, \cdots$ converges to b.

PROOF Convergence $z_n \to c = a + ib$ implies convergence $x_n \to a$ and $y_n \to b$ because if $|z_n - c| < \epsilon$, then z_n lies within the circle of radius ϵ about $c = a + ib$, so that (Fig. 363a)

$$|x_n - a| < \epsilon, \qquad |y_n - b| < \epsilon.$$

Conversely, if $x_n \to a$ and $y_n \to b$ as $n \to \infty$, then for a given $\epsilon > 0$ we can choose N so large that, for every $n > N$,

$$|x_n - a| < \frac{\epsilon}{2}, \qquad |y_n - b| < \frac{\epsilon}{2}.$$

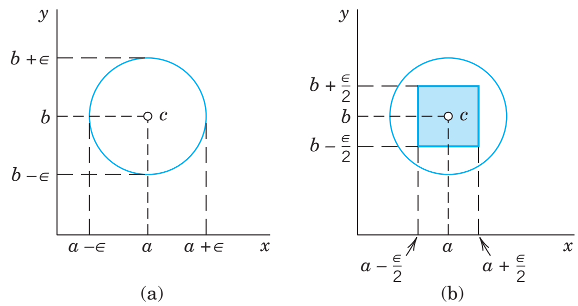


Fig. 363. Proof of Theorem 1

These two inequalities imply that $z_n = x_n + iy_n$ lies in a square with center c and side ϵ. Hence, z_n must lie within a circle of radius ϵ with center c (Fig. 363b). ■

Series

Given a sequence $z_1, z_2, \cdots, z_m, \cdots$, we may form the sequence of the sums

$$s_1 = z_1, \qquad s_2 = z_1 + z_2, \qquad s_3 = z_1 + z_2 + z_3, \quad \cdots$$

and in general

$$s_n = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \cdots). \tag{2}$$

Here s_n is called the ***n*th partial sum** of the *infinite series* or **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots. \tag{3}$$

The $z_1, z_2, \cdots$ are called the **terms** of the series. (Our usual ***summation letter*** is n, unless we need n for another purpose, as here, and we then use m as the summation letter.)

A **convergent series** is one whose sequence of partial sums converges, say,

$$\lim_{n\to\infty} s_n = s. \qquad \text{Then we write} \qquad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

and call s the **sum** or *value* of the series. A series that is not convergent is called a **divergent series**.

If we omit the terms of s_n from (3), there remains

$$R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots. \tag{4}$$

This is called the **remainder** *of the series* (3) *after the term* z_n. Clearly, if (3) converges and has the sum s, then

$$s = s_n + R_n, \qquad \text{thus} \qquad R_n = s - s_n.$$

Now $s_n \to s$ by the definition of convergence; hence $R_n \to 0$. In applications, when s is unknown and we compute an approximation s_n of s, then $|R_n|$ is the error, and $R_n \to 0$ means that we can make $|R_n|$ as small as we please, by choosing n large enough.

An application of Theorem 1 to the partial sums immediately relates the convergence of a complex series to that of the two series of its real parts and of its imaginary parts:

THEOREM 2

Real and Imaginary Parts

A series (3) with $z_m = x_m + iy_m$ converges and has the sum $s = u + iv$ if and only if $x_1 + x_2 + \cdots$ converges and has the sum u and $y_1 + y_2 + \cdots$ converges and has the sum v.

Tests for Convergence and Divergence of Series

Convergence tests in complex are practically the same as in calculus. We apply them before we use a series, to make sure that the series converges.

Divergence can often be shown very simply as follows.

THEOREM 3

Divergence

If a series $z_1 + z_2 + \cdots$ converges, then $\lim_{m\to\infty} z_m = 0$. Hence if this does not hold, the series diverges.

PROOF If $z_1 + z_2 + \cdots$ converges, with the sum s, then, since $z_m = s_m - s_{m-1}$,

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty}(s_m - s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = s - s = 0. \quad \blacksquare$$

CAUTION! $z_m \to 0$ is *necessary* for convergence but *not sufficient*, as we see from the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which satisfies this condition but diverges, as is shown in calculus (see, for example, Ref. [GenRef11] in App. 1).

The practical difficulty in proving convergence is that, in most cases, the sum of a series is unknown. Cauchy overcame this by showing that a series converges if and only if its partial sums eventually get close to each other:

THEOREM 4

Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \cdots$ is convergent if and only if for every given $\epsilon > 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that

$$(5) \qquad |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots$$

The somewhat involved proof is left optional (see App. 4).

Absolute Convergence. A series $z_1 + z_2 + \cdots$ is called **absolutely convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$

is convergent.

If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, **conditionally convergent**.

EXAMPLE 3 A Conditionally Convergent Series

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots$ converges, but only conditionally since the harmonic series diverges, as mentioned above (after Theorem 3). ■

If a series is absolutely convergent, it is convergent.

This follows readily from Cauchy's principle (see Prob. 29). This principle also yields the following general convergence test.

THEOREM 5

Comparison Test

If a series $z_1 + z_2 + \cdots$ is given and we can find a convergent series $b_1 + b_2 + \cdots$ with nonnegative real terms such that $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$, then the given series converges, even absolutely.

PROOF By Cauchy's principle, since $b_1 + b_2 + \cdots$ converges, for any given $\epsilon > 0$ we can find an N such that

$$b_{n+1} + \cdots + b_{n+p} < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots.$$

From this and $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$ we conclude that for those n and p,

$$|z_{n+1}| + \cdots + |z_{n+p}| \leqq b_{n+1} + \cdots + b_{n+p} < \epsilon.$$

Hence, again by Cauchy's principle, $|z_1| + |z_2| + \cdots$ converges, so that $z_1 + z_2 + \cdots$ is absolutely convergent. ■

A good comparison series is the geometric series, which behaves as follows.

THEOREM 6

Geometric Series

The ***geometric series***

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots \tag{6*}$$

converges with the sum $1/(1 - q)$ *if* $|q| < 1$ *and diverges if* $|q| \geqq 1$.

PROOF If $|q| \geqq 1$, then $|q^m| \geqq 1$ and Theorem 3 implies divergence.

Now let $|q| < 1$. The nth partial sum is

$$s_n = 1 + q + \cdots + q^n.$$

From this,

$$qs_n = \qquad q + \cdots + q^n + q^{n+1}.$$

On subtraction, most terms on the right cancel in pairs, and we are left with

$$s_n - qs_n = (1 - q)s_n = 1 - q^{n+1}.$$

Now $1 - q \neq 0$ since $q \neq 1$, and we may solve for s_n, finding

$$s_n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q}. \tag{6}$$

Since $|q| < 1$, the last term approaches zero as $n \to \infty$. Hence if $|q| < 1$, the series is convergent and has the sum $1/(1 - q)$. This completes the proof. ■

Ratio Test

This is the most important test in our further work. We get it by taking the geometric series as comparison series $b_1 + b_2 + \cdots$ in Theorem 5:

THEOREM 7

Ratio Test

If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ has the property that for every n greater than some N,

$$\left|\frac{z_{n+1}}{z_n}\right| \leqq q < 1 \qquad (n > N) \tag{7}$$

(where $q < 1$ is fixed), this series converges absolutely. If for every $n > N$,

$$\left|\frac{z_{n+1}}{z_n}\right| \geqq 1 \qquad (n > N), \tag{8}$$

the series diverges.

PROOF If (8) holds, then $|z_{n+1}| \geqq |z_n|$ for $n > N$, so that divergence of the series follows from Theorem 3.

If (7) holds, then $|z_{n+1}| \leqq |z_n|\, q$ for $n > N$, in particular,

$$|z_{N+2}| \leqq |z_{N+1}|q, \qquad |z_{N+3}| \leqq |z_{N+2}|q \leqq |z_{N+1}|q^2, \qquad \text{etc.,}$$

and in general, $|z_{N+p}| \leqq |z_{N+1}|q^{p-1}$. Since $q < 1$, we obtain from this and Theorem 6

$$|z_{N+1}| + |z_{N+2}| + |z_{N+3}| + \cdots \leqq |z_{N+1}|\,(1 + q + q^2 + \cdots) \leqq |z_{N+1}|\,\frac{1}{1 - q}.$$

Absolute convergence of $z_1 + z_2 + \cdots$ now follows from Theorem 5. ■

CAUTION! The inequality (7) implies $|z_{n+1}/z_n| < 1$, but this does ***not*** imply convergence, as we see from the harmonic series, which satisfies $z_{n+1}/z_n = n/(n+1) < 1$ for all n but diverges.

If the sequence of the ratios in (7) and (8) converges, we get the more convenient

THEOREM 8

Ratio Test

If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ is such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, *then:*

(a) *If $L < 1$, the series converges absolutely.*

(b) *If $L > 1$, the series diverges.*

(c) *If $L = 1$, the series may converge or diverge, so that the test fails and permits no conclusion.*

PROOF **(a)** We write $k_n = |z_{n+1}/z_n|$ and let $L = 1 - b < 1$. Then by the definition of limit, the k_n must eventually get close to $1 - b$, say, $k_n \leqq q = 1 - \frac{1}{2}b < 1$ for all n greater than some N. Convergence of $z_1 + z_2 + \cdots$ now follows from Theorem 7.

(b) Similarly, for $L = 1 + c > 1$ we have $k_n \geqq 1 + \frac{1}{2}c > 1$ for all $n > N^*$ (sufficiently large), which implies divergence of $z_1 + z_2 + \cdots$ by Theorem 7.

(c) The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ has $z_{n+1}/z_n = n/(n+1)$, hence $L = 1$, and diverges. The series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \qquad \text{has} \qquad \frac{z_{n+1}}{z_n} = \frac{n^2}{(n+1)^2},$$

hence also $L = 1$, but it converges. Convergence follows from (Fig. 364)

$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leqq 1 + \int_1^n \frac{dx}{x^2} = 2 - \frac{1}{n},$$

so that $s_1, s_2, \cdots$ is a bounded sequence and is monotone increasing (since the terms of the series are all positive); both properties together are sufficient for the convergence of the real sequence $s_1, s_2, \cdots$. (In calculus this is proved by the so-called *integral test*, whose idea we have used.) ■

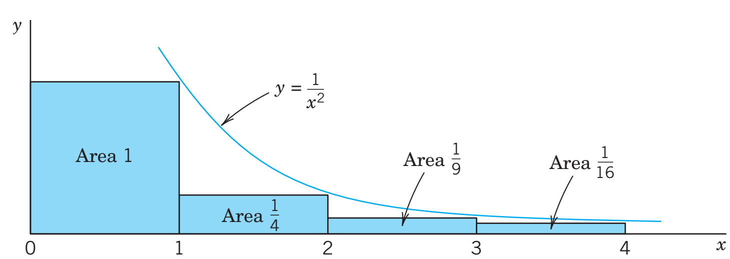


Fig. 364. Convergence of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

EXAMPLE 4 **Ratio Test**

Is the following series convergent or divergent? (First guess, then calculate.)

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Solution. By Theorem 8, the series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}/(n+1)!}{|100+75i|^n/n!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \quad\to\quad L = 0.$$ ■

EXAMPLE 5 **Theorem 7 More General Than Theorem 8**

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Solution. The ratios of the absolute values of successive terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$. Hence convergence follows from Theorem 7. Since the sequence of these ratios has no limit, Theorem 8 is not applicable. ■

Root Test

The ratio test and the root test are the two practically most important tests. The ratio test is usually simpler, but the root test is somewhat more general.

THEOREM 9 **Root Test**

If a series $z_1 + z_2 + \cdots$ is such that for every n greater than some N,

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N) \tag{9}$$

(where $q < 1$ is fixed), this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geqq 1, \tag{10}$$

the series diverges.

PROOF If (9) holds, then $|z_n| \leqq q^n < 1$ for all $n > N$. Hence the series $|z_1| + |z_2| + \cdots$ converges by comparison with the geometric series, so that the series $z_1 + z_2 + \cdots$ converges absolutely. If (10) holds, then $|z_n| \geqq 1$ for infinitely many n. Divergence of $z_1 + z_2 + \cdots$ now follows from Theorem 3. ■

CAUTION! Equation (9) implies $\sqrt[n]{|z_n|} < 1$, but this does not imply convergence, as we see from the harmonic series, which satisfies $\sqrt[n]{1/n} < 1$ (for $n > 1$) but diverges.

If the sequence of the roots in (9) and (10) converges, we more conveniently have

THEOREM 10

Root Test

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:

(a) *The series converges absolutely if $L < 1$.*

(b) *The series diverges if $L > 1$.*

(c) *If $L = 1$, the test fails; that is, no conclusion is possible.*

PROBLEM SET 15.1

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1 + i)^{2n}/2^n$

2. $z_n = (3 + 4i)^n/n!$

3. $z_n = n\pi/(4 + 2ni)$

4. $z_n = (1 + 2i)^n$

5. $z_n = (-1)^n + 10i$

6. $z_n = (\cos n\pi i)/n$

7. $z_n = n^2 + i/n^2$

8. $z_n = [(1 + 3i)/\sqrt{10}]^n$

9. $z_n = (3 + 3i)^{-n}$

10. $z_n = \sin(\frac{1}{4}n\pi) + i^n$

11. CAS EXPERIMENT. Sequences. Write a program for graphing complex sequences. Use the program to discover sequences that have interesting "geometric" properties, e.g., lying on an ellipse, spiraling to its limit, having infinitely many limit points, etc.

12. Addition of sequences. If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ is convergent with the limit $l + l^*$.

13. Bounded sequence. Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

14. On Theorem 1. Illustrate Theorem 1 by an example of your own.

15. On Theorem 2. Give another example illustrating Theorem 2.

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\sum_{n=0}^{\infty} \frac{(20 + 30i)^n}{n!}$

17. $\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

18. $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{4}\right)^n$

19. $\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

20. $\sum_{n=0}^{\infty} \frac{n + i}{3n^2 + 2i}$

21. $\sum_{n=0}^{\infty} \frac{(\pi + \pi i)^{2n+1}}{(2n + 1)!}$

22. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

23. $\sum_{n=0}^{\infty} \frac{(-1)^n(1 + i)^{2n}}{(2n)!}$

24. $\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$

25. $\sum_{n=1}^{\infty} \frac{i^n}{n}$

26. Significance of (7). What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?

27. On Theorems 7 and 8. Give another example showing that Theorem 7 is more general than Theorem 8.

28. CAS EXPERIMENT. Series. Write a program for computing and graphing numeric values of the first n partial sums of a series of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.

29. Absolute convergence. Show that if a series converges absolutely, it is convergent.

30. Estimate of remainder. Let $|z_{n+1}/z_n| \leqq q < 1$, so that the series $z_1 + z_2 + \cdots$ converges by the ratio test. Show that the remainder $R_n = z_{n+1} + z_{n+2} + \cdots$ satisfies the inequality $|R_n| \leqq |z_{n+1}|/(1 - q)$. Using this, find how many terms suffice for computing the sum s of the series

$$\sum_{n=1}^{\infty} \frac{n + i}{2^n n}$$

with an error not exceeding 0.05 and compute s to this accuracy.

15.2 Power Series

The student should pay close attention to the material because we shall show how power series play an important role in complex analysis. Indeed, they are the most important series in complex analysis because their sums are analytic functions (Theorem 5, Sec. 15.3), and every analytic function can be represented by power series (Theorem 1, Sec. 15.4).

A **power series** *in powers of* $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \tag{1}$$

where z is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the **coefficients** of the series, and z_0 is a complex (or real) constant, called the **center** of the series. This generalizes real power series of calculus.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* z:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots. \tag{2}$$

Convergence Behavior of Power Series

Power series have variable terms (functions of z), but ***if we fix z, then all the concepts for series with constant terms in the last section apply***. Usually a series with variable terms will converge for some z and diverge for others. For a power series the situation is simple. The series (1) may converge in a disk with center z_0 or in the whole z-plane or only at z_0. We illustrate this with typical examples and then prove it.

EXAMPLE 1 Convergence in a Disk. Geometric Series

The *geometric series*

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geqq 1$ (see Theorem 6 in Sec. 15.1). ■

EXAMPLE 2 Convergence for Every z

The power series (which will be the Maclaurin series of e^z in Sec. 15.4)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is absolutely convergent for every z. In fact, by the ratio test, for any fixed z,

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \quad \text{as} \quad n \to \infty. \quad ■$$

EXAMPLE 3 **Convergence Only at the Center. (Useless Series)**

The following power series converges only at $z = 0$, but diverges for every $z \neq 0$, as we shall show.

$$\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

In fact, from the ratio test we have

$$\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = (n+1)|z| \quad \longrightarrow \quad \infty \qquad \text{as} \qquad n \longrightarrow \infty \qquad (z \text{ fixed and } \neq 0).$$ ■

THEOREM 1 **Convergence of a Power Series**

(a) *Every power series* (1) *converges at the center* z_0.

(b) *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 365.

(c) *If* (1) *diverges at* $z = z_2$, *it diverges for every* z *farther away from* z_0 *than* z_2. See Fig. 365.

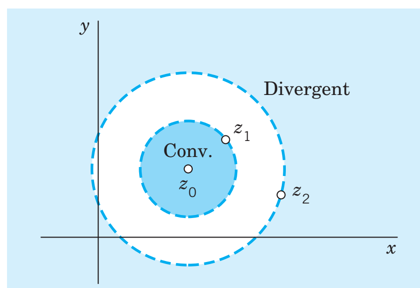


Fig. 365. Theroem 1

PROOF **(a)** For $z = z_0$ the series reduces to the single term a_0.

(b) Convergence at $z = z_1$ gives by Theorem 3 in Sec. 15.1 $a_n(z_1 - z_0)^n \longrightarrow 0$ as $n \longrightarrow \infty$. This implies boundedness in absolute value,

$$|a_n(z_1 - z_0)^n| < M \qquad \text{for every } n = 0, 1, \cdots.$$

Multiplying and dividing $a_n(z - z_0)^n$ by $(z_1 - z_0)^n$ we obtain from this

$$|a_n(z - z_0)^n| = \left|a_n(z_1 - z_0)^n \left(\frac{z - z_0}{z_1 - z_0}\right)^n\right| \leqq M \left|\frac{z - z_0}{z_1 - z_0}\right|^n.$$

Summation over n gives

$$\sum_{n=1}^{\infty} |a_n(z - z_0)^n| \leqq M \sum_{n=1}^{\infty} \left|\frac{z - z_0}{z_1 - z_0}\right|^n. \tag{3}$$

Now our assumption $|z - z_0| < |z_1 - z_0|$ implies that $|(z - z_0)/(z_1 - z_0)| < 1$. Hence the series on the right side of (3) is a converging geometric series (see Theorem 6 in

Sec. 15.1). Absolute convergence of (1) as stated in (b) now follows by the comparison test in Sec. 15.1.

(c) If this were false, we would have convergence at a z_3 farther away from z_0 than z_2. This would imply convergence at z_2, by (b), a contradiction to our assumption of divergence at z_2. ■

Radius of Convergence of a Power Series

Convergence for every z (the nicest case, Example 2) or for no $z \neq z_0$ (the useless case, Example 3) needs no further discussion, and we put these cases aside for a moment. We consider the ***smallest*** circle with center z_0 that includes all the points at which a given power series (1) converges. Let R denote its radius. The circle

$$|z - z_0| = R \qquad \text{(Fig. 366)}$$

is called the **circle of convergence** and its radius R the **radius of convergence** of (1). Theorem 1 then implies convergence everywhere within that circle, that is, for all z for which

$$|z - z_0| < R \tag{4}$$

(the open disk with center z_0 and radius R). Also, since R is as *small* as possible, the series (1) diverges for all z for which

$$|z - z_0| > R. \tag{5}$$

No general statements can be made about the convergence of a power series (1) ***on the circle of convergence*** itself. The series (1) may converge at some or all or none of the points. Details will not be important to us. Hence a simple example may just give us the idea.

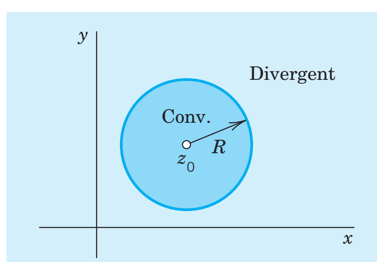


Fig. 366. Circle of convergence

EXAMPLE 4 **Behavior on the Circle of Convergence**

On the circle of convergence (radius $R = 1$ in all three series),

$\Sigma z^n/n^2$ converges everywhere since $\Sigma 1/n^2$ converges,

$\Sigma z^n/n$ converges at -1 (by Leibniz's test) but diverges at 1,

Σz^n diverges everywhere. ■

Notations $R = \infty$ and $R = 0$. To incorporate these two excluded cases in the present notation, we write

$R = \infty$ if the series (1) converges for all z (as in Example 2),
$R = 0$ if (1) converges only at the center $z = z_0$ (as in Example 3).

These are convenient notations, but nothing else.

Real Power Series. In this case in which powers, coefficients, and center are real, formula (4) gives the **convergence interval** $|x - x_0| < R$ of length $2R$ on the real line.

Determination of the Radius of Convergence from the Coefficients. For this important practical task we can use

THEOREM 2

Radius of Convergence R

Suppose that the sequence $|a_{n+1}/a_n|$, $n = 1, 2, \cdots$, *converges with limit* L^*. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* z. *If* $L^* \neq 0$ (*hence* $L^* > 0$), *then*

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad (6)$$

(Cauchy–Hadamard formula[1]**).**

If $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ (*convergence only at the center* z_0).

PROOF For (1) the ratio of the terms in the ratio test (Sec. 15.1) is

$$\left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z - z_0|. \qquad \text{The limit is} \qquad L = L^*|z - z_0|.$$

Let $L^* \neq 0$, thus $L^* > 0$. We have convergence if $L = L^*|z - z_0| < 1$, thus $|z - z_0| < 1/L^*$, and divergence if $|z - z_0| > 1/L^*$. By (4) and (5) this shows that $1/L^*$ is the convergence radius and proves (6).

If $L^* = 0$, then $L = 0$ for every z, which gives convergence for all z by the ratio test. If $|a_{n+1}/a_n| \to \infty$, then $|a_{n+1}/a_n||z - z_0| > 1$ for any $z \neq z_0$ and all sufficiently large n. This implies divergence for all $z \neq z_0$ by the ratio test (Theorem 7, Sec. 15.1). ■

Formula (6) will not help if L^* does not exist, but extensions of Theorem 2 are still possible, as we discuss in Example 6 below.

EXAMPLE 5

Radius of Convergence

By (6) the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z - 3i)^n$ is

$$R = \lim_{n\to\infty} \left[\frac{(2n!)}{(n!)^2} \bigg/ \frac{(2n+2)!}{((n+1)!)^2} \right] = \lim_{n\to\infty} \left[\frac{(2n!)}{(2n+2)!} \cdot \frac{((n+1)!)^2}{(n!)^2} \right] = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

The series converges in the open disk $|z - 3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$. ■

[1]Named after the French mathematicians A. L. CAUCHY (see Sec. 2.5) and JACQUES HADAMARD (1865–1963). Hadamard made basic contributions to the theory of power series and devoted his lifework to partial differential equations.

EXAMPLE 6 Extension of Theorem 2

Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

Solution. The sequence of the ratios $\frac{1}{6}$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem 2 is of no help. It can be shown that

$$R = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}. \tag{6*}$$

This still does not help here, since $(\sqrt[n]{|a_n|})$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = \frac{1}{2}$ for odd n, whereas for even n we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \quad \text{as} \quad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $\frac{1}{2}$ and 1. It can further be shown that

$$R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}. \tag{6**}$$

Here $\tilde{l} = I$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ■

Summary. Power series converge in an open circular disk or some even for every z (or some only at the center, but they are useless); for the radius of convergence, see (6) or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance in complex analysis.

PROBLEM SET 15.2

1. **Power series.** Are $1/z + z + z^2 + \cdots$ and $z + z^{3/2} + z^2 + z^3 + \cdots$ power series? Explain.
2. **Radius of convergence.** What is it? Its role? What motivates its name? How can you find it?
3. **Convergence.** What are the only basically different possibilities for the convergence of a power series?
4. **On Examples 1–3.** Extend them to power series in powers of $z - 4 + 3\pi i$. Extend Example 1 to the case of radius of convergence 6.
5. **Powers z^{2n}.** Show that if $\Sigma a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma a_n z^{2n}$ has radius of convergence $\sqrt{R}$.

6–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence.

6. $\sum_{n=0}^{\infty} 4^n(z+1)^n$
7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{1}{2}\pi\right)^{2n}$
8. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z-\pi i)^n$
9. $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$
10. $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$
11. $\sum_{n=0}^{\infty} \left(\frac{2-i}{1+5i}\right)z^n$
12. $\sum_{n=0}^{\infty} \frac{(-1)^n n}{8^n} z^n$
13. $\sum_{n=0}^{\infty} 16^n(z+i)^{4n}$
14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$
15. $\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2}(z-2i)^n$
16. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n(n!)^3} z^n$
17. $\sum_{n=1}^{\infty} \frac{2^n}{n(n+1)} z^{2n+1}$
18. $\sum_{n=0}^{\infty} \frac{2(-1)^n}{\sqrt{\pi}(2n+1)n!} z^{2n+1}$
19. **CAS PROJECT. Radius of Convergence.** Write a program for computing R from (6), (6*), or (6**), in

this order, depending on the existence of the limits needed. Test the program on some series of your choice such that all three formulas (6), (6*), and (6**) will come up.

20. TEAM PROJECT. Radius of Convergence.

(a) Understanding (6). Formula (6) for R contains $|a_n/a_{n+1}|$, not $|a_{n+1}/a_n|$. How could you memorize this by using a qualitative argument?

(b) Change of coefficients. What happens to R $(0 < R < \infty)$ if you (i) multiply all a_n by $k \neq 0$, (ii) multiply all a_n by $k^n \neq 0$, (iii) replace a_n by $1/a_n$? Can you think of an application of this?

(c) Understanding Example 6, which extends Theorem 2 to nonconvergent cases of a_n/a_{n+1}. Do you understand the principle of "mixing" by which Example 6 was obtained? Make up further examples.

(d) Understanding (b) and (c) in Theorem 1. Does there exist a power series in powers of z that converges at $z = 30 + 10i$ and diverges at $z = 31 - 6i$? Give reason.

15.3 Functions Given by Power Series

Here, our main goal is to show that power series represent analytic functions. This fact (Theorem 5) and the fact that power series behave nicely under addition, multiplication, differentiation, and integration accounts for their usefulness.

To simplify the formulas in this section, we take $z_0 = 0$ and write

$$\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

There is no loss of generality because a series in powers of $\hat{z} - z_0$ with any z_0 can always be reduced to the form (1) if we set $\hat{z} - z_0 = z$.

Terminology and Notation. If any given power series (1) has a nonzero radius of convergence R (thus $R > 0$), its sum is a function of z, say $f(z)$. Then we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

We say that $f(z)$ *is* **represented** *by the power series* or that *it is* **developed** *in the power series.* For instance, the geometric series *represents* the function $f(z) = 1/(1 - z)$ in the interior of the unit circle $|z| = 1$. (See Theorem 6 in Sec. 15.1.)

Uniqueness of a Power Series Representation. This is our next goal. It means that *a function $f(z)$ cannot be represented by two different power series with the same center.* We claim that if $f(z)$ can at all be developed in a power series with center z_0, the development is unique. This important fact is frequently used in complex analysis (as well as in calculus). We shall prove it in Theorem 2. The proof will follow from

THEOREM 1

Continuity of the Sum of a Power Series

If a function $f(z)$ can be represented by a power series (2) *with radius of convergence $R > 0$, then $f(z)$ is continuous at $z = 0$.*

PROOF From (2) with $z = 0$ we have $f(0) = a_0$. Hence by the definition of continuity we must show that $\lim_{z\to 0} f(z) = f(0) = a_0$. That is, we must show that for a given $\epsilon > 0$ there is a $\delta > 0$ such that $|z| < \delta$ implies $|f(z) - a_0| < \epsilon$. Now (2) converges absolutely for $|z| \leqq r$ with any r such that $0 < r < R$, by Theorem 1 in Sec. 15.2. Hence the series

$$\sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} |a_n| r^n$$

converges. Let $S \neq 0$ be its sum. ($S = 0$ is trivial.) Then for $0 < |z| \leqq r$,

$$|f(z) - a_0| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leqq |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1} \leqq |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z| S$$

and $|z|S < \epsilon$ when $|z| < \delta$, where $\delta > 0$ is less than r and less than ϵ/S. Hence $|z|S < \delta S < (\epsilon/S)S = \epsilon$. This proves the theorem. ■

From this theorem we can now readily obtain the desired uniqueness theorem (again assuming $z_0 = 0$ without loss of generality):

THEOREM 2

Identity Theorem for Power Series. Uniqueness

Let the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ and $b_0 + b_1 z + b_2 z^2 + \cdots$ both be convergent for $|z| < R$, where R is positive, and let them both have the same sum for all these z. Then the series are identical, that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, $\cdots$.

*Hence if a function $f(z)$ can be represented by a power series with any center z_0, this representation is **unique**.*

PROOF We proceed by induction. By assumption,

$$a_0 + a_1 z + a_2 z^2 + \cdots = b_0 + b_1 z + b_2 z^2 + \cdots \qquad (|z| < R).$$

The sums of these two power series are continuous at $z = 0$, by Theorem 1. Hence if we consider $|z| > 0$ and let $z \to 0$ on both sides, we see that $a_0 = b_0$: the assertion is true for $n = 0$. Now assume that $a_n = b_n$ for $n = 0, 1, \cdots, m$. Then on both sides we may omit the terms that are equal and divide the result by z^{m+1} ($\neq 0$); this gives

$$a_{m+1} + a_{m+2} z + a_{m+3} z^2 + \cdots = b_{m+1} + b_{m+2} z + b_{m+3} z^2 + \cdots.$$

Similarly as before by letting $z \to 0$ we conclude from this that $a_{m+1} = b_{m+1}$. This completes the proof. ■

Operations on Power Series

Interesting in itself, this discussion will serve as a preparation for our main goal, namely, to show that functions represented by power series are analytic.

Termwise addition or subtraction of two power series with radii of convergence R_1 and R_2 yields a power series with radius of convergence at least equal to the smaller of R_1 and R_2. *Proof.* Add (or subtract) the partial sums s_n and s_n^* term by term and use $\lim (s_n \pm s_n^*) = \lim s_n \pm \lim s_n^*$.

Termwise multiplication of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \cdots$$

and

$$g(z) = \sum_{m=0}^{\infty} b_m z^m = b_0 + b_1 z + \cdots$$

means the multiplication of each term of the first series by each term of the second series and the collection of like powers of z. This gives a power series, which is called the **Cauchy product** of the two series and is given by

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) z^n.$$

We mention without proof that this power series converges absolutely for each z within the smaller circle of convergence of the two given series and has the sum $s(z) = f(z)g(z)$. For a proof, see [D5] listed in App. 1.

Termwise differentiation and integration of power series is permissible, as we show next. We call **derived series** *of the power series* (1) the power series obtained from (1) by termwise differentiation, that is,

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots. \tag{3}$$

THEOREM 3

Termwise Differentiation of a Power Series

The derived series of a power series has the same radius of convergence as the original series.

PROOF This follows from (6) in Sec. 15.2 because

$$\lim_{n\to\infty} \frac{n|a_n|}{(n+1)|a_{n+1}|} = \lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

or, if the limit does not exist, from (6**) in Sec. 15.2 by noting that $\sqrt[n]{n} \to 1$ as $n \to \infty$. ■

EXAMPLE 1 **Application of Theorem 3**

Find the radius of convergence R of the following series by applying Theorem 3.

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n = z^2 + 3z^3 + 6z^4 + 10z^5 + \cdots.$$

Solution. Differentiate the geometric series twice term by term and multiply the result by $z^2/2$. This yields the given series. Hence $R = 1$ by Theorem 3. ■

THEOREM 4 **Termwise Integration of Power Series**

The power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ *term by term has the same radius of convergence as the original series.*

The proof is similar to that of Theorem 3.

With the help of Theorem 3, we establish the main result in this section.

Power Series Represent Analytic Functions

THEOREM 5 **Analytic Functions. Their Derivatives**

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.

PROOF **(a)** We consider any power series (1) with positive radius of convergence R. Let $f(z)$ be its sum and $f_1(z)$ the sum of its derived series; thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{4}$$

We show that $f(z)$ is analytic and has the derivative $f_1(z)$ in the interior of the circle of convergence. We do this by proving that for any fixed z with $|z| < R$ and $\Delta z \to 0$ the difference quotient $[f(z + \Delta z) - f(z)]/\Delta z$ approaches $f_1(z)$. By termwise addition we first have from (4)

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left[\frac{(z + \Delta z)^n - z^n}{\Delta z} - n z^{n-1} \right]. \tag{5}$$

Note that the summation starts with 2, since the constant term drops out in taking the difference $f(z + \Delta z) - f(z)$, and so does the linear term when we subtract $f_1(z)$ from the difference quotient.

(b) We claim that the series in (5) can be written

$$\sum_{n=2}^{\infty} a_n \,\Delta z[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-2)z^{n-3}(z + \Delta z) + (n-1)z^{n-2}]. \qquad (6)$$

The somewhat technical proof of this is given in App. 4.

(c) We consider (6). The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $(n-1)^2 \leqq n(n-1)$, we see that for $|z| \leqq R_0$ and $|z + \Delta z| \leqq R_0$, $R_0 < R$, the absolute value of this series (6) cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n| n(n-1) R_0^{n-2}. \qquad (7)$$

This series with a_n instead of $|a_n|$ is the second derived series of (2) at $z = R_0$ and converges absolutely by Theorem 3 of this section and Theorem 1 of Sec. 15.2. Hence our present series (7) converges. Let the sum of (7) (without the factor $|\Delta z|$) be $K(R_0)$. Since (6) is the right side of (5), our present result is

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) \right| \leqq |\Delta z| K(R_0).$$

Letting $\Delta z \to 0$ and noting that R_0 $(< R)$ is arbitrary, we conclude that $f(z)$ is analytic at any point interior to the circle of convergence and its derivative is represented by the derived series. From this the statements about the higher derivatives follow by induction. ■

Summary. The results in this section show that power series are about as nice as we could hope for: we can differentiate and integrate them term by term (Theorems 3 and 4). Theorem 5 accounts for the great importance of power series in complex analysis: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus in turn are analytic functions. But this is only part of the story. In the next section we show that, conversely, *every* given analytic function $f(z)$ can be represented by power series, called ***Taylor series*** and being the complex analog of the real Taylor series of calculus.

PROBLEM SET 15.3

1. **Relation to Calculus.** Material in this section generalizes calculus. Give details.
2. **Termwise addition.** Write out the details of the proof on termwise addition and subtraction of power series.
3. **On Theorem 3.** Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$, as claimed.
4. **Cauchy product.** Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n+1)z^n$

   (a) by using the Cauchy product, **(b)** by differentiating a suitable series.

5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: **(a)** directly by the Cauchy–Hadamard formula in Sec. 15.2, and **(b)** from a series of simpler terms by using Theorem 3 or Theorem 4.

5. $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z - 2i)^n$
6. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$
7. $\sum_{n=1}^{\infty} \frac{n}{3^n}(z + 2i)^{2n}$
8. $\sum_{n=1}^{\infty} \frac{5^n}{n(n+1)} z^n$

9. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)} z^{2n}$

10. $\sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{z}{2}\right)^n$

11. $\sum_{n=1}^{\infty} \frac{3^n n(n+1)}{7^n} (z+2)^{2n}$

12. $\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$

13. $\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

14. $\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

15. $\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n} (z-i)^n$

16–20 APPLICATIONS OF THE IDENTITY THEOREM

State clearly and explicitly where and how you are using Theorem 2.

16. **Even functions.** If $f(z)$ in (2) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd n. Give examples.

17. **Odd function.** If $f(z)$ in (2) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even n. Give examples.

18. **Binomial coefficients.** Using $(1+z)^p(1+z)^q = (1+z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

19. Find applications of Theorem 2 in differential equations and elsewhere.

20. **TEAM PROJECT. Fibonacci numbers.**[2] **(a)** The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence (a_{n+1}/a_n).

(b) Fibonacci's rabbit problem. Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

(c) Generating function. Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

15.4 Taylor and Maclaurin Series

The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{1}$$

or, by (1), Sec. 14.4,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*. \tag{2}$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180–1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

[3] BROOK TAYLOR (1685–1731), English mathematician who introduced real Taylor series. COLIN MACLAURIN (1698–1746), Scots mathematician, professor at Edinburgh.

The **remainder** of the Taylor series (1) after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \, dz^* \tag{3}$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

$$\begin{aligned} f(z) = f(z_0) &+ \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots \\ &+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z). \end{aligned} \tag{4}$$

This is called **Taylor's formula** *with remainder.*

We see that ***Taylor series are power series***. From the last section we know that power series represent analytic functions. And we now show that *every* analytic function can be represented by power series, namely, by Taylor series (with various centers). This makes Taylor series very important in complex analysis. Indeed, they are more fundamental in complex analysis than their real counterparts are in calculus.

THEOREM 1

Taylor's Theorem

Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor series (1) *with center z_0 that represents $f(z)$. This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainders $R_n(z)$ of* (1) *can be represented in the form* (3). *The coefficients satisfy the inequality*

$$|a_n| \leqq \frac{M}{r^n} \tag{5}$$

where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

PROOF The key tool is Cauchy's integral formula in Sec. 14.3; writing z and z^* instead of z_0 and z (so that z^* is the variable of integration), we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} \, dz^*. \tag{6}$$

z lies inside C, for which we take a circle of radius r with center z_0 and interior in D (Fig. 367). We develop $1/(z^* - z)$ in (6) in powers of $z - z_0$. By a ***standard algebraic manipulation*** (worth remembering!) we first have

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \dfrac{z - z_0}{z^* - z_0}\right)}. \tag{7}$$

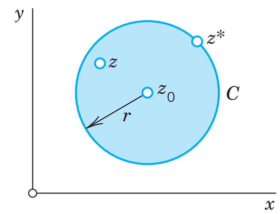


Fig. 367. Cauchy formula (6)

For later use we note that since z^* is on C while z is inside C, we have

$$\left|\frac{z - z_0}{z^* - z_0}\right| < 1. \qquad \text{(Fig. 367).} \tag{7*}$$

To (7) we now apply the sum formula for a finite geometric sum

$$1 + q + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \qquad (q \neq 1), \tag{8*}$$

which we use in the form (take the last term to the other side and interchange sides)

$$\frac{1}{1 - q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1 - q}. \tag{8}$$

Applying this with $q = (z - z_0)/(z^* - z_0)$ to the right side of (7), we get

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right]$$
$$+ \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}.$$

We insert this into (6). Powers of $z - z_0$ do not depend on the variable of integration z^*, so that we may take them out from under the integral sign. This yields

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0}\,dz^* + \frac{z - z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^2}\,dz^* + \cdots$$
$$\cdots + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\,dz^* + R_n(z)$$

with $R_n(z)$ given by (3). The integrals are those in (2) related to the derivatives, so that we have proved the Taylor formula (4).

Since analytic functions have derivatives of all orders, we can take n in (4) as large as we please. If we let n approach infinity, we obtain (1). Clearly, (1) will converge and represent $f(z)$ if and only if

$$\lim_{n\to\infty} R_n(z) = 0. \tag{9}$$

We prove (9) as follows. Since z^* lies on C, whereas z lies inside C (Fig. 367), we have $|z^* - z| > 0$. Since $f(z)$ is analytic inside and on C, it is bounded, and so is the function $f(z^*)/(z^* - z)$, say,

$$\left| \frac{f(z^*)}{z^* - z} \right| \leqq \widetilde{M}$$

for all z^* on C. Also, C has the radius $r = |z^* - z_0|$ and the length $2\pi r$. Hence by the *ML*-inequality (Sec. 14.1) we obtain from (3)

$$\begin{aligned} |R_n| &= \frac{|z - z_0|^{n+1}}{2\pi} \left| \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^* \right| \\ &\leqq \frac{|z - z_0|^{n+1}}{2\pi} \widetilde{M} \frac{1}{r^{n+1}} 2\pi r = \widetilde{M} \left| \frac{z - z_0}{r} \right|^{n+1}. \end{aligned} \tag{10}$$

Now $|z - z_0| < r$ because z lies *inside* C. Thus $|z - z_0|/r < 1$, so that the right side approaches 0 as $n \to \infty$. This proves that the Taylor series converges and has the sum $f(z)$. Uniqueness follows from Theorem 2 in the last section. Finally, (5) follows from a_n in (1) and the Cauchy inequality in Sec. 14.4. This proves Taylor's theorem. ■

Accuracy of Approximation. We can achieve any preassinged accuracy in approximating $f(z)$ by a partial sum of (1) by choosing n large enough. This is the practical use of formula (9).

Singularity, Radius of Convergence. On the circle of convergence of (1) there is at least one **singular point** of $f(z)$, that is, a point $z = c$ at which $f(z)$ is not analytic (but such that every disk with center c contains points at which $f(z)$ *is* analytic). We also say that $f(z)$ **is singular** at c or **has a singularity** at c. Hence the radius of convergence R of (1) is usually equal to the distance from z_0 to the nearest singular point of $f(z)$.

(Sometimes R can be greater than that distance: Ln z is singular on the negative real axis, whose distance from $z_0 = -1 + i$ is 1, but the Taylor series of Ln z with center $z_0 = -1 + i$ has radius of convergence $\sqrt{2}$.)

Power Series as Taylor Series

Taylor series are power series—of course! Conversely, we have

THEOREM 2 **Relation to the Previous Section**

A power series with a nonzero radius of convergence is the Taylor series of its sum.

PROOF Given the power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots .$$

Then $f(z_0) = a_0$. By Theorem 5 in Sec. 15.3 we obtain

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots, \qquad \text{thus} \qquad f'(z_0) = a_1$$
$$f''(z) = 2a_2 + 3 \cdot 2(z - z_0) + \cdots, \qquad \text{thus} \qquad f''(z_0) = 2!a_2$$

and in general $f^{(n)}(z_0) = n!a_n$. With these coefficients the given series becomes the Taylor series of $f(z)$ with center z_0. ■

Comparison with Real Functions. One surprising property of complex analytic functions is that they have derivatives of all orders, and now we have discovered the other surprising property that they can always be represented by power series of the form (1). This is not true in general for ***real functions;*** there are real functions that have derivatives of all orders but cannot be represented by a power series. (Example: $f(x) = \exp(-1/x^2)$ if $x \neq 0$ and $f(0) = 0$; this function cannot be represented by a Maclaurin series in an open disk with center 0 because all its derivatives at 0 are zero.)

Important Special Taylor Series

These are as in calculus, with x replaced by complex z. Can you see why? (*Answer.* The coefficient formulas are the same.)

EXAMPLE 1 Geometric Series

Let $f(z) = 1/(1 - z)$. Then we have $f^{(n)}(z) = n!/(1 - z)^{n+1}, f^{(n)}(0) = n!$. Hence the Maclaurin expansion of $1/(1 - z)$ is the geometric series

(11)
$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1).$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence. ■

EXAMPLE 2 Exponential Function

We know that the exponential function e^z (Sec. 13.5) is analytic for all z, and $(e^z)' = e^z$. Hence from (1) with $z_0 = 0$ we obtain the Maclaurin series

(12)
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots .$$

This series is also obtained if we replace x in the familiar Maclaurin series of e^x by z.

Furthermore, by setting $z = iy$ in (12) and separating the series into the real and imaginary parts (see Theorem 2, Sec. 15.1) we obtain

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} .$$

Since the series on the right are the familiar Maclaurin series of the real functions $\cos y$ and $\sin y$, this shows that we have rediscovered the **Euler formula**

(13)
$$e^{iy} = \cos y + i \sin y.$$

Indeed, one may use (12) for ***defining*** e^z and derive from (12) the basic properties of e^z. For instance, the differentiation formula $(e^z)' = e^z$ follows readily from (12) by termwise differentiation. ■

EXAMPLE 3 Trigonometric and Hyperbolic Functions

By substituting (12) into (1) of Sec. 13.6 we obtain

(14)
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots .$$

When $z = x$ these are the familiar Maclaurin series of the real functions $\cos x$ and $\sin x$. Similarly, by substituting (12) into (11), Sec. 13.6, we obtain

(15)
$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$
$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots .$$
■

EXAMPLE 4 Logarithm

From (1) it follows that

(16)
$$\text{Ln}\,(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \qquad (|z| < 1).$$

Replacing z by $-z$ and multiplying both sides by -1, we get

(17)
$$-\text{Ln}\,(1 - z) = \text{Ln}\,\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (|z| < 1).$$

By adding both series we obtain

(18)
$$\text{Ln}\,\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1).$$
■

Practical Methods

The following examples show ways of obtaining Taylor series more quickly than by the use of the coefficient formulas. Regardless of the method used, the result will be the same. This follows from the uniqueness (see Theorem 1).

EXAMPLE 5 Substitution

Find the Maclaurin series of $f(z) = 1/(1 + z^2)$.

Solution. By substituting $-z^2$ for z in (11) we obtain

(19)
$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad (|z| < 1).$$
■

EXAMPLE 6 **Integration**

Find the Maclaurin series of $f(z) = \arctan z$.

Solution. We have $f'(z) = 1/(1 + z^2)$. Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$ defined as that value for which $|u| < \pi/2$.

EXAMPLE 7 **Development by Using the Geometric Series**

Develop $1/(c - z)$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

Solution. This was done in the proof of Theorem 1, where $c = z^*$. The beginning was simple algebra and then the use of (11) with z replaced by $(z - z_0)/(c - z_0)$:

$$\frac{1}{c-z} = \frac{1}{c - z_0 - (z - z_0)} = \frac{1}{(c - z_0)\left(1 - \dfrac{z - z_0}{c - z_0}\right)} = \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n$$

$$= \frac{1}{c - z_0}\left(1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots\right).$$

This series converges for

$$\left|\frac{z - z_0}{c - z_0}\right| < 1, \qquad \text{that is,} \qquad |z - z_0| < |c - z_0|.$$

EXAMPLE 8 **Binomial Series, Reduction by Partial Fractions**

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Solution. We develop $f(z)$ in partial fractions and the first fraction in a **binomial series**

$$\text{(20)} \qquad \frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$
$$= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

with $m = 2$ and the second fraction in a geometric series, and then add the two series term by term. This gives

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{[3 + (z-1)]^2} - \frac{2}{2 - (z-1)} = \frac{1}{9}\left(\frac{1}{[1 + \frac{1}{3}(z-1)]^2}\right) - \frac{1}{1 - \frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots.$$

We see that the first series converges for $|z - 1| < 3$ and the second for $|z - 1| < 2$. This had to be expected because $1/(z+2)^2$ is singular at -2 and $2/(z-3)$ at 3, and these points have distance 3 and 2, respectively, from the center $z_0 = 1$. Hence the whole series converges for $|z - 1| < 2$.

PROBLEM SET 15.4

1. **Calculus.** Which of the series in this section have you discussed in calculus? What is new?
2. **On Examples 5 and 6.** Give all the details in the derivation of the series in those examples.

3–10 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence.

3. $\sin 2z^2$
4. $\dfrac{z+2}{1-z^2}$
5. $\dfrac{1}{2+z^4}$
6. $\dfrac{1}{1+3iz}$
7. $\cos^2 \tfrac{1}{2}z$
8. $\sin^2 z$
9. $\displaystyle\int_0^z \exp\left(\frac{-t^2}{2}\right) dt$
10. $\displaystyle\exp(z^2)\int_0^z \exp(-t^2)\, dt$

11–14 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf z, **sine integral** $\mathrm{Si}(z)$, and **Fresnel integrals**[4] $S(z)$ and $C(z)$, which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

11. $\displaystyle S(z) = \int_0^z \sin t^2\, dt$
12. $\displaystyle C(z) = \int_0^z \cos t^2\, dt$
13. $\displaystyle \mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\, dt$
14. $\displaystyle \mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\, dt$

15. **CAS Project. sec, tan. (a) Euler numbers.** The Maclaurin series

$$\sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots \tag{21}$$

defines the *Euler numbers* E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the E_{2n} from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GenRef1], p. 810, listed in App. 1.)

(b) Bernoulli numbers. The Maclaurin series

$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots \tag{22}$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that

$$\begin{aligned} B_1 &= -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0,\\ B_4 &= -\tfrac{1}{30}, \quad B_5 = 0, \quad B_6 = \tfrac{1}{42}, \cdots. \end{aligned} \tag{23}$$

Write a program for computing B_n.

(c) Tangent. Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$\begin{aligned} \tan z &= \frac{2i}{e^{2iz}-1} - \frac{4i}{e^{4iz}-1} - i \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} z^{2n-1}. \end{aligned} \tag{24}$$

16. **Inverse sine.** Developing $1/\sqrt{1-z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \quad (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

17. **TEAM PROJECT. Properties from Maclaurin Series.** Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

(a) The formulas for the derivatives of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$. and $\mathrm{Ln}(1+z)$

(b) $\tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

(c) $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

18–25 TAYLOR SERIES

Find the Taylor series with center z_0 and its radius of convergence.

18. $1/z, \quad z_0 = i$
19. $1/(1-z), \quad z_0 = i$
20. $\cos^2 z, \quad z_0 = \pi/2$
21. $\sin z, \quad z_0 = \pi/2$
22. $\cosh(z - \pi i), \quad z_0 = \pi i$
23. $1/(z+i)^2, \quad z_0 = i$
24. $e^{z(z-2)}, \quad z_0 = 1$
25. $\sinh(2z - i), \quad z_0 = i/2$

[4] AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics.

15.5 Uniform Convergence. *Optional*

We know that power series are *absolutely convergent* (Sec. 15.2, Theorem 1) and, as another basic property, we now show that they are *uniformly convergent.* Since uniform convergence is of general importance, for instance, in connection with termwise integration of series, we shall discuss it quite thoroughly.

To define uniform convergence, we consider a series whose terms are any complex functions $f_0(z), f_1(z), \cdots$

$$\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + f_2(z) + \cdots. \tag{1}$$

(This includes power series as a special case in which $f_m(z) = a_m(z - z_0)^m$.) We assume that the series (1) converges for all z in some region G. We call its sum $s(z)$ and its nth partial sum $s_n(z)$; thus

$$s_n(z) = f_0(z) + f_1(z) + \cdots + f_n(z).$$

Convergence in G means the following. If we pick a $z = z_1$ in G, then, by the definition of convergence at z_1, for given $\epsilon > 0$ we can find an $N_1(\epsilon)$ such that

$$|s(z_1) - s_n(z_1)| < \epsilon \qquad \text{for all } n > N_1(\epsilon).$$

If we pick a z_2 in G, keeping ϵ as before, we can find an $N_2(\epsilon)$ such that

$$|s(z_2) - s_n(z_2)| < \epsilon \qquad \text{for all } n > N_2(\epsilon),$$

and so on. Hence, given an $\epsilon > 0$, to each z in G there corresponds a number $N_z(\epsilon)$. This number tells us how many terms we need (what s_n we need) at a z to make $|s(z) - s_n(z)|$ smaller than ϵ. Thus this number $N_z(\epsilon)$ measures the speed of convergence.

Small $N_z(\epsilon)$ means rapid convergence, large $N_z(\epsilon)$ means slow convergence at the point z considered. Now, if we can find an $N(\epsilon)$ larger than all these $N_z(\epsilon)$ for all z in G, we say that the convergence of the series (1) in G is *uniform.* Hence this basic concept is defined as follows.

DEFINITION

Uniform Convergence

A series (1) with sum $s(z)$ is called **uniformly convergent** in a region G if for every $\epsilon > 0$ we can find an $N = N(\epsilon)$, ***not depending on z***, such that

$$|s(z) - s_n(z)| < \epsilon \qquad \text{for all } n > N(\epsilon) \textbf{\textit{ and all z in G}}.$$

Uniformity of convergence is thus a property that always refers to an *infinite set* in the z-plane, that is, a set consisting of infinitely many points.

EXAMPLE 1 **Geometric Series**

Show that the geometric series $1 + z + z^2 + \cdots$ is (a) uniformly convergent in any closed disk $|z| \leqq r < 1$, (b) not uniformly convergent in its whole disk of convergence $|z| < 1$.

Solution. **(a)** For z in that closed disk we have $|1 - z| \geqq 1 - r$ (sketch it). This implies that $1/|1 - z| \leqq 1/(1 - r)$. Hence (remember (8) in Sec. 15.4 with $q = z$)

$$|s(z) - s_n(z)| = \left| \sum_{m=n+1}^{\infty} z^m \right| = \left| \frac{z^{n+1}}{1 - z} \right| \leqq \frac{r^{n+1}}{1 - r}.$$

Since $r < 1$, we can make the right side as small as we want by choosing n large enough, and since the right side does not depend on z (in the closed disk considered), this means that the convergence is uniform.

(b) For given real K (no matter how large) and n we can always find a z in the disk $|z| < 1$ such that

$$\left| \frac{z^{n+1}}{1 - z} \right| = \frac{|z|^{n+1}}{|1 - z|} > K,$$

simply by taking z close enough to 1. Hence no single $N(\epsilon)$ will suffice to make $|s(z) - s_n(z)|$ smaller than a given $\epsilon > 0$ *throughout the whole disk.* By definition, this shows that the convergence of the geometric series in $|z| < 1$ is not uniform. ■

This example suggests that *for a power series, the uniformity of convergence may at most be disturbed near the circle of convergence.* This is true:

THEOREM 1

Uniform Convergence of Power Series

A power series

$$\sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{2}$$

with a nonzero radius of convergence R is uniformly convergent in every circular disk $|z - z_0| \leqq r$ of radius $r < R$.

PROOF For $|z - z_0| \leqq r$ and any positive integers n and p we have

$$|a_{n+1}(z - z_0)^{n+1} + \cdots + a_{n+p}(z - z_0)^{n+p}| \leqq |a_{n+1}| r^{n+1} + \cdots + |a_{n+p}| r^{n+p}. \tag{3}$$

Now (2) converges absolutely if $|z - z_0| = r < R$ (by Theorem 1 in Sec. 15.2). Hence it follows from the Cauchy convergence principle (Sec. 15.1) that, an $\epsilon > 0$ being given, we can find an $N(\epsilon)$ such that

$$|a_{n+1}| r^{n+1} + \cdots + |a_{n+p}| r^{n+p} < \epsilon \qquad \text{for } n > N(\epsilon) \quad \text{and} \quad p = 1, 2, \cdots.$$

From this and (3) we obtain

$$|a_{n+1}(z - z_0)^{n+1} + \cdots + a_{n+p}(z - z_0)^{n+p}| < \epsilon$$

for all z in the disk $|z - z_0| \leqq r$, every $n > N(\epsilon)$, and every $p = 1, 2, \cdots$. Since $N(\epsilon)$ is independent of z, this shows uniform convergence, and the theorem is proved. ■

Thus we have established uniform convergence of power series, the basic concern of this section. *We now shift from power series to arbitary series of variable terms* and examine uniform convergence in this more general setting. This will give a deeper understanding of uniform convergence.

Properties of Uniformly Convergent Series

Uniform convergence derives its main importance from two facts:

1. If a series of *continuous* terms is uniformly convergent, its sum is also continuous (Theorem 2, below).

2. Under the same assumptions, termwise integration is permissible (Theorem 3).

This raises two questions:

1. How can a converging series of continuous terms manage to have a discontinuous sum? (Example 2)

2. How can something go wrong in termwise integration? (Example 3)

Another natural question is:

3. What is the relation between absolute convergence and uniform convergence? The surprising answer: none. (Example 5)

These are the ideas we shall discuss.

If we add *finitely many* continuous functions, we get a continuous function as their sum. Example 2 will show that this is no longer true for an infinite series, even if it converges absolutely. However, if it converges *uniformly,* this cannot happen, as follows.

THEOREM 2

Continuity of the Sum

Let the series

$$\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$$

be uniformly convergent in a region G. Let $F(z)$ be its sum. Then if each term $f_m(z)$ is continuous at a point z_1 in G, the function $F(z)$ is continuous at z_1.

PROOF Let $s_n(z)$ be the nth partial sum of the series and $R_n(z)$ the corresponding remainder:

$$s_n = f_0 + f_1 + \cdots + f_n, \qquad R_n = f_{n+1} + f_{n+2} + \cdots.$$

Since the series converges uniformly, for a given $\epsilon > 0$ we can find an $N = N(\epsilon)$ such that

$$|R_N(z)| < \frac{\epsilon}{3} \qquad \text{for all } z \text{ in } G.$$

Since $s_N(z)$ is a sum of finitely many functions that are continuous at z_1, this sum is continuous at z_1. Therefore, we can find a $\delta > 0$ such that

$$|s_N(z) - s_N(z_1)| < \frac{\epsilon}{3} \quad \text{for all } z \text{ in } G \text{ for which } |z - z_1| < \delta.$$

Using $F = s_N + R_N$ and the triangle inequality (Sec. 13.2), for these z we thus obtain

$$\begin{aligned} |F(z) - F(z_1)| &= |s_N(z) + R_N(z) - [s_N(z_1) + R_N(z_1)]| \\ &\leqq |s_N(z) - s_N(z_1)| + |R_N(z)| + |R_N(z_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This implies that $F(z)$ is continuous at z_1, and the theorem is proved. ■

EXAMPLE 2 Series of Continuous Terms with a Discontinuous Sum

Consider the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots \qquad (x \text{ real}).$$

This is a geometric series with $q = 1/(1 + x^2)$ times a factor x^2. Its nth partial sum is

$$s_n(x) = x^2\left[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n}\right].$$

We now use the trick by which one finds the sum of a geometric series, namely, we multiply $s_n(x)$ by $-q = -1/(1 + x^2)$,

$$-\frac{1}{1+x^2}s_n(x) = -x^2\left[\frac{1}{1+x^2} + \cdots + \frac{1}{(1+x^2)^n} + \frac{1}{(1+x^2)^{n+1}}\right].$$

Adding this to the previous formula, simplifying on the left, and canceling most terms on the right, we obtain

$$\frac{x^2}{1+x^2}s_n(x) = x^2\left[1 - \frac{1}{(1+x^2)^{n+1}}\right],$$

thus

$$s_n(x) = 1 + x^2 - \frac{1}{(1+x^2)^n}.$$

The exciting Fig. 368 "explains" what is going on. We see that if $x \neq 0$, the sum is

$$s(x) = \lim_{n\to\infty} s_n(x) = 1 + x^2,$$

but for $x = 0$ we have $s_n(0) = 1 - 1 = 0$ for all n, hence $s(0) = 0$. So we have the surprising fact that the sum is discontinuous (at $x = 0$), although all the terms are continuous and the series converges even absolutely (its terms are nonnegative, thus equal to their absolute value!).

Theorem 2 now tells us that the convergence cannot be uniform in an interval containing $x = 0$. We can also verify this directly. Indeed, for $x \neq 0$ the remainder has the absolute value

$$|R_n(x)| = |s(x) - s_n(x)| = \frac{1}{(1+x^2)^n}$$

and we see that for a given ϵ (<1) we cannot find an N depending only on ϵ such that $|R_n| < \epsilon$ for all $n > N(\epsilon)$ and all x, say, in the interval $0 \leqq x \leqq 1$. ■

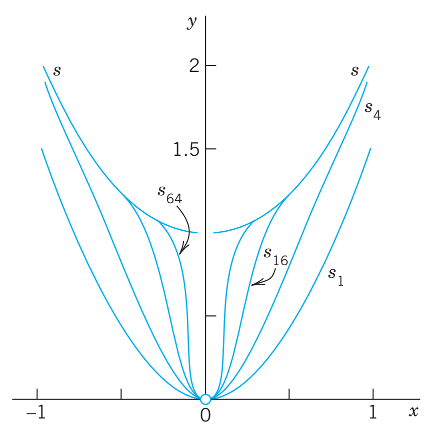


Fig. 368. Partial sums in Example 2

Termwise Integration

This is our second topic in connection with uniform convergence, and we begin with an example to become aware of the danger of just blindly integrating term-by-term.

EXAMPLE 3 **Series for Which Termwise Integration Is Not Permissible**

Let $u_m(x) = mxe^{-mx^2}$ and consider the series

$$\sum_{m=0}^{\infty} f_m(x) \qquad \text{where} \qquad f_m(x) = u_m(x) - u_{m-1}(x)$$

in the interval $0 \leqq x \leqq 1$. The nth partial sum is

$$s_n = u_1 - u_0 + u_2 - u_1 + \cdots + u_n - u_{n-1} = u_n - u_0 = u_n.$$

Hence the series has the sum $F(x) = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} u_n(x) = 0 \quad (0 \leqq x \leqq 1)$. From this we obtain

$$\int_0^1 F(x)\,dx = 0.$$

On the other hand, by integrating term by term and using $f_1 + f_2 + \cdots + f_n = s_n$, we have

$$\sum_{m=1}^{\infty} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \sum_{m=1}^{n} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \int_0^1 s_n(x)\,dx.$$

Now $s_n = u_n$ and the expression on the right becomes

$$\lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \lim_{n\to\infty} \int_0^1 nxe^{-nx^2}\,dx = \lim_{n\to\infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2},$$

but not 0. This shows that the series under consideration cannot be integrated term by term from $x = 0$ to $x = 1$. ∎

The series in Example 3 is not uniformly convergent in the interval of integration, and we shall now prove that in the case of a uniformly convergent series of continuous functions we may integrate term by term.

THEOREM 3 **Termwise Integration**

Let

$$F(z) = \sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$$

be a uniformly convergent series of continuous functions in a region G. Let C be any path in G. Then the series

$$\sum_{m=0}^{\infty} \int_C f_m(z)\,dz = \int_C f_0(z)\,dz + \int_C f_1(z)\,dz + \cdots \tag{4}$$

is convergent and has the sum $\int_C F(z)\,dz$.

PROOF From Theorem 2 it follows that $F(z)$ is continuous. Let $s_n(z)$ be the nth partial sum of the given series and $R_n(z)$ the corresponding remainder. Then $F = s_n + R_n$ and by integration,

$$\int_C F(z)\,dz = \int_C s_n(z)\,dz + \int_C R_n(z)\,dz.$$

Let L be the length of C. Since the given series converges uniformly, for every given $\epsilon > 0$ we can find a number N such that $|R_n(z)| < \epsilon/L$ for all $n > N$ and all z in G. By applying the ML-inequality (Sec. 14.1) we thus obtain

$$\left| \int_C R_n(z)\, dz \right| < \frac{\epsilon}{L} L = \epsilon \qquad \text{for all } n > N.$$

Since $R_n = F - s_n$, this means that

$$\left| \int_C F(z)\, dz - \int_C s_n(z)\, dz \right| < \epsilon \qquad \text{for all } n > N.$$

Hence, the series (4) converges and has the sum indicated in the theorem. ■

Theorems 2 and 3 characterize the two most important properties of uniformly convergent series. Also, since differentiation and integration are inverse processes, Theorem 3 implies

THEOREM 4

Termwise Differentiation

Let the series $f_0(z) + f_1(z) + f_2(z) + \cdots$ be convergent in a region G and let $F(z)$ be its sum. Suppose that the series $f_0'(z) + f_1'(z) + f_2'(z) + \cdots$ converges uniformly in G and its terms are continuous in G. Then

$$F'(z) = f_0'(z) + f_1'(z) + f_2'(z) + \cdots \qquad \textit{for all } z \textit{ in } G.$$

Test for Uniform Convergence

Uniform convergence is usually proved by the following comparison test.

THEOREM 5

Weierstrass[5] *M*-Test for Uniform Convergence

Consider a series of the form (1) *in a region G of the z-plane. Suppose that one can find a convergent series of constant terms,*

$$M_0 + M_1 + M_2 + \cdots, \tag{5}$$

such that $|f_m(z)| \leqq M_m$ for all z in G and every $m = 0, 1, \cdots$. Then (1) *is uniformly convergent in G.*

The simple proof is left to the student (Team Project 18).

---

[5]KARL WEIERSTRASS (1815–1897), great German mathematician, who developed complex analysis based on the concept of power series and residue integration. (See footnote in Section 13.4.) He put analysis on a sound theoretical footing. His mathematical rigor is so legendary that one speaks *Weierstrassian rigor*. (See paper by Birkhoff and Kreyszig, 1984 in footnote in Sec. 5.5; Kreyszig, E., On the Calculus, of Variations and Its Major Influences on the Mathematics of the First Half of Our Century. Part II, *American Mathematical Monthly* (1994), 101, No. 9, pp. 902–908). Weierstrass also made contributions to the calculus of variations, approximation theory, and differential geometry. He obtained the concept of uniform convergence in 1841 (published 1894, *sic!*); the first publication on the concept was by G. G. STOKES (see Sec 10.9) in 1847.

EXAMPLE 4 **Weierstrass *M*-Test**

Does the following series converge uniformly in the disk $|z| \leqq 1$?

$$\sum_{m=1}^{\infty} \frac{z^m + 1}{m^2 + \cosh m|z|}.$$

Solution. Uniform convergence follows by the Weierstrass M-test and the convergence of $\Sigma 1/m^2$ (see Sec. 15.1, in the proof of Theorem 8) because

$$\left| \frac{z^m + 1}{m^2 + \cosh m|z|} \right| \leqq \frac{|z|^m + 1}{m^2} \leqq \frac{2}{m^2}.$$

■

No Relation Between Absolute and Uniform Convergence

We finally show the surprising fact that there are series that converge absolutely but not uniformly, and others that converge uniformly but not absolutely, so that there is no relation between the two concepts.

EXAMPLE 5 **No Relation Between Absolute and Uniform Convergence**

The series in Example 2 converges absolutely but not uniformly, as we have shown. On the other hand, the series

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{x^2 + m} = \frac{1}{x^2 + 1} - \frac{1}{x^2 + 2} + \frac{1}{x^2 + 3} - + \cdots \qquad (x \text{ real})$$

converges uniformly on the whole real line but not absolutely.

Proof. By the familiar Leibniz test of calculus (see App. A3.3) the remainder R_n does not exceed its first term in absolute value, since we have a series of alternating terms whose absolute values form a monotone decreasing sequence with limit zero. Hence given $\epsilon > 0$, for all x we have

$$|R_n(x)| \leqq \frac{1}{x^2 + n + 1} < \frac{1}{n} < \epsilon \qquad \text{if } n > N(\epsilon) \geqq \frac{1}{\epsilon}.$$

This proves uniform convergence, since $N(\epsilon)$ does not depend on x.

The convergence is not absolute because for any *fixed* x we have

$$\left| \frac{(-1)^{m-1}}{x^2 + m} \right| = \frac{1}{x^2 + m} > \frac{k}{m}$$

where k is a suitable constant, and $k\Sigma 1/m$ diverges. ■

PROBLEM SET 15.5

1. **CAS EXPERIMENT. Graphs of Partial Sums. (a) Fig. 368.** Produce this exciting figure using your CAS. Add further curves, say, those of s_{256}, s_{1024}, etc. on the same screen.

   (b) Power series. Study the nonuniformity of convergence experimentally by graphing partial sums near the endpoints of the convergence interval for real $z = x$.

2–9 POWER SERIES

Where does the power series converge uniformly? Give reason.

2. $\sum_{n=0}^{\infty} \left(\frac{n+2}{7n-3}\right)^n z^n$

3. $\sum_{n=0}^{\infty} \frac{1}{3^n}(z+i)^{2n}$

4. $\sum_{n=0}^{\infty} \frac{3^n(1-i)^n}{n!}(z-i)^n$

5. $\sum_{n=2}^{\infty} \binom{n}{2}(4z+2i)^n$

6. $\sum_{n=0}^{\infty} 2^n(\tanh n^2) z^{2n}$

7. $\sum_{n=1}^{\infty} \frac{n!}{n^2}\left(z+\frac{1}{2}i\right)$

8. $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)}(z-1)^{2n}$

9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2}(z-2i)^n$

10–17 UNIFORM CONVERGENCE

Prove that the series converges uniformly in the indicated region.

10. $\sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}, \quad |z| \leqq 10^{20}$

11. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \leqq 1$

12. $\sum_{n=1}^{\infty} \frac{z^n}{n^3 \cosh n|z|}, \quad |z| \leqq 1$

13. $\sum_{n=1}^{\infty} \frac{\sin^n |z|}{n^2}, \quad \text{all } z$

14. $\sum_{n=0}^{\infty} \frac{z^n}{|z|^{2n}+1}, \quad 2 \leqq |z| \leqq 10$

15. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n!)} z^n, \quad |z| \leqq 3$

16. $\sum_{n=1}^{\infty} \frac{\tanh^n |z|}{n(n+1)}, \quad \text{all } z$

17. $\sum_{n=1}^{\infty} \frac{\pi^n}{n^4} z^{2n}, \quad |z| \leqq 0.56$

18. TEAM PROJECT. Uniform Convergence.

(a) Weierstrass *M*-test. Give a proof.

(b) Termwise differentiation. Derive Theorem 4 from Theorem 3.

(c) Subregions. Prove that uniform convergence of a series in a region G implies uniform convergence in any portion of G. Is the converse true?

(d) Example 2. Find the precise region of convergence of the series in Example 2 with x replaced by a complex variable z.

(e) Figure 369. Show that $x^2 \sum_{m=1}^{\infty} (1+x^2)^{-m} = 1$ if $x \neq 0$ and 0 if $x = 0$. Verify by computation that the partial sums s_1, s_2, s_3 look as shown in Fig. 369.

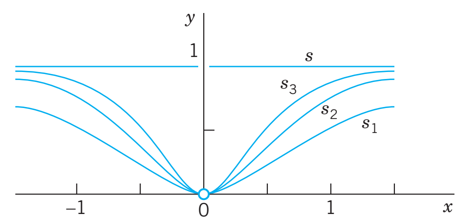


Fig. 369. Sum s and partial sums in Team Project 18(e)

19–20 HEAT EQUATION

Show that (9) in Sec. 12.6 with coefficients (10) is a solution of the heat equation for $t > 0$, assuming that $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and has one-sided derivatives at all interior points of that interval. Proceed as follows.

19. Show that $|B_n|$ is bounded, say $|B_n| < K$ for all n. Conclude that

$$|u_n| < Ke^{-\lambda_n^2 t_0} \qquad \text{if} \qquad t \geqq t_0 > 0$$

and, by the Weierstrass test, the series (9) converges uniformly with respect to x and t for $t \geqq t_0$, $0 \leqq x \leqq L$. Using Theorem 2, show that $u(x, t)$ is continuous for $t \geqq t_0$ and thus satisfies the boundary conditions (2) for $t \geqq t_0$.

20. Show that $|\partial u_n/\partial t| < \lambda_n^2 K e^{-\lambda_n^2 t_0}$ if $t \geqq t_0$ and the series of the expressions on the right converges, by the ratio test. Conclude from this, the Weierstrass test, and Theorem 4 that the series (9) can be differentiated term by term with respect to t and the resulting series has the sum $\partial u/\partial t$. Show that (9) can be differentiated twice with respect to x and the resulting series has the sum $\partial^2 u/\partial x^2$. Conclude from this and the result to Prob. 19 that (9) is a solution of the heat equation for all $t \geqq t_0$. (The proof that (9) satisfies the given initial condition can be found in Ref. [C10] listed in App. 1.)

CHAPTER 15 REVIEW QUESTIONS AND PROBLEMS

1. What is convergence test for series? State two tests from memory. Give examples.
2. What is a power series? Why are these series very important in complex analysis?
3. What is absolute convergence? Conditional convergence? Uniform convergence?
4. What do you know about convergence of power series?
5. What is a Taylor series? Give some basic examples.
6. What do you know about adding and multiplying power series?
7. Does every function have a Taylor series development? Explain.
8. Can properties of functions be discovered from Maclaurin series? Give examples.
9. What do you know about termwise integration of series?
10. How did we obtain Taylor's formula from Cauchy's formula?

11–15 RADIUS OF CONVERGENCE

Find the radius of convergence.

11. $\sum_{n=2}^{\infty} \frac{n+1}{n^2+1}(z+1)^n$

12. $\sum_{n=2}^{\infty} \frac{4^n}{n-1}(z-\pi i)^n$

13. $\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n}(z-i)^n$

14. $\sum_{n=1}^{\infty} \frac{n^5}{n!}(z-3i)^{2n}$

15. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{2n} z^n$

16–20 RADIUS OF CONVERGENCE

Find the radius of convergence. Try to identify the sum of the series as a familiar function.

16. $\sum_{n=1}^{\infty} \frac{z^n}{n}$

17. $\sum_{n=0}^{\infty} \frac{z^n}{n!} z^n$

18. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\pi z)^{2n+1}$

19. $\sum_{n=0}^{\infty} \frac{z^n}{(2n)!}$

20. $\sum_{n=0}^{\infty} \frac{z^n}{(3+4i)^n}$

21–25 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence. Show details.

21. $(\sinh z^2)/z^2$

22. $1/(1-z)^3$

23. $\cos^2 z$

24. $1/(\pi z+1)$

25. $-(\exp/(-z^2)-1)/z^2$

26–30 TAYLOR SERIES

Find the Taylor series with the given point as center and its radius of convergence.

26. $z^4, \quad i$

27. $\cos z, \quad \frac{1}{2}\pi$

28. $1/z, \quad 2i$

29. $\operatorname{Ln} z, \quad 3$

30. $e^z, \quad \pi i$

SUMMARY OF CHAPTER 15
Power Series, Taylor Series

Sequences, series, and convergence tests are discussed in Sec. 15.1. A **power series** is of the form (Sec. 15.2)

$$(1) \qquad \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots;$$

z_0 is its *center*. The series (1) converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$, where R is the **radius of convergence**. Some power series converge for all z (then we write $R = \infty$). In exceptional cases a power series may converge only at the center; such a series is practically useless. Also, $R = \lim |a_n/a_{n+1}|$ if this limit exists. The series (1) converges absolutely (Sec. 15.2) and ***uniformly*** (Sec. 15.5) in every closed disk $|z - z_0| \leqq r < R$ $(R > 0)$. It represents an analytic function $f(z)$ for $|z - z_0| < R$. The derivatives $f'(z), f''(z), \cdots$ are obtained by termwise differentiation of (1), and these series have the same radius of convergence R as (1). See Sec. 15.3.

Conversely, *every* analytic function $f(z)$ can be represented by power series. These **Taylor series** of $f(z)$ are of the form (Sec. 15.4)

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \qquad (|z - z_0| < R), \tag{2}$$

as in calculus. They converge for all z in the open disk with center z_0 and radius generally equal to the distance from z_0 to the nearest **singularity** of $f(z)$ (point at which $f(z)$ ceases to be analytic as defined in Sec. 15.4). If $f(z)$ is **entire** (analytic for all z; see Sec. 13.5), then (2) converges for all z. The functions e^z, $\cos z$, $\sin z$, etc. have Maclaurin series, that is, Taylor series with center 0, similar to those in calculus (Sec. 15.4).

CHAPTER 16

Laurent Series. Residue Integration

The main purpose of this chapter is to learn about another powerful method for evaluating complex integrals and certain real integrals. It is called *residue integration.* Recall that the first method of evaluating complex integrals consisted of directly applying Cauchy's integral formula of Sec. 14.3. Then we learned about Taylor series (Chap. 15) and will now generalize Taylor series. The beauty of residue integration, the second method of integration, is that it brings together a lot of the previous material.

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disk, a ***Laurent series*** (Sec. 16.1) is a series of positive ***and negative*** integer powers of $z - z_0$ and converges in an annulus (a circular ring) with center z_0. Hence, by a Laurent series, we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

We know that for a given function the Taylor series with a given center z_0 is unique. We shall see that, in contrast, a function $f(z)$ can have several Laurent series with the same center z_0 and valid in several concentric annuli. The most important of these series is the one that converges for $0 < |z - z_0| < R$, that is, everywhere near the center z_0 except at z_0 itself, where z_0 is a singular point of $f(z)$. The series (or finite sum) of the negative powers of *this* Laurent series is called the **principal part** of the singularity of $f(z)$ at z_0, and is used to classify this singularity (Sec. 16.2). The coefficient of the power $1/(z - z_0)$ of *this* series is called the **residue** of $f(z)$ at z_0. Residues are used in an elegant and powerful integration method, called ***residue integration***, for complex contour integrals (Sec. 16.3) as well as for certain complicated real integrals (Sec. 16.4).

Prerequisite: Chaps. 13, 14, Sec. 15.2.
Sections that may be omitted in a shorter course: 16.2, 16.4.
References and Answers to Problems: App. 1 Part D, App. 2.

16.1 Laurent Series

Laurent series generalize Taylor series. If, in an application, we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at z_0 (as defined in Sec. 15.4), we cannot use a Taylor series. Instead we can use a new kind of series, called **Laurent series**,[1]

[1]PIERRE ALPHONSE LAURENT (1813–1854), French military engineer and mathematician, published the theorem in 1843.

consisting of positive integer powers of $z - z_0$ (and a constant) as well as ***negative integer powers*** of $z - z_0$; this is the new feature.

Laurent series are also used for classifying singularities (Sec. 16.2) and in a powerful integration method ("residue integration," Sec. 16.3).

A Laurent series of $f(z)$ converges in an annulus (in the "hole" of which $f(z)$ may have singularities), as follows.

THEOREM 1

Laurent's Theorem

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them (blue in Fig. 370). *Then $f(z)$ can be represented by the Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1}$$
$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$
$$\cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals

$$(2) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*)\, dz^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle, as in Fig. 370. [The variable of integration is denoted by z^* since z is used in (1).]

This series converges and represents $f(z)$ in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where $f(z)$ is singular.

In the important special case that z_0 is the only singular point of $f(z)$ inside C_2, this circle can be shrunk to the point z_0, giving convergence in a disk except at the center. In this case the series (or finite sum) of the negative powers of (1) *is called the* **principal part** *of $f(z)$ at z_0* [*or of that Laurent series* (1)].

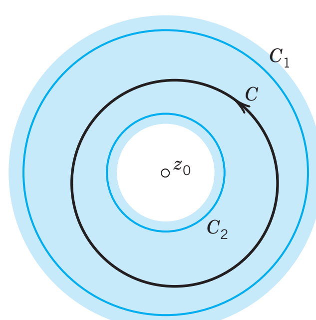


Fig. 370. Laurent's theorem

COMMENT. Obviously, instead of (1), (2) we may write (denoting b_n by a_{-n})

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \tag{1'}$$

where all the coefficients are now given by a single integral formula, namely,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{2'}$$

Let us now prove Laurent's theorem.

PROOF **(a)** ***The nonnegative powers*** are those of a Taylor series.

To see this, we use Cauchy's integral formula (3) in Sec. 14.3 with z^* (instead of z) as the variable of integration and z instead of z_0. Let $g(z)$ and $h(z)$ denote the functions represented by the two terms in (3), Sec. 14.3. Then

$$f(z) = g(z) + h(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z}\, dz^* - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^* - z}\, dz^*. \tag{3}$$

Here z is any point in the given annulus and we integrate counterclockwise over both C_1 and C_2, so that the minus sign appears since in (3) of Sec. 14.3 the integration over C_2 is taken clockwise. We transform each of these two integrals as in Sec. 15.4. The first integral is precisely as in Sec. 15.4. Hence we get exactly the same result, namely, the Taylor series of $g(z)$,

$$g(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z}\, dz^* = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{4}$$

with coefficients [see (2), Sec. 15.4, counterclockwise integration]

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*. \tag{5}$$

Here we can replace C_1 by C (see Fig. 370), by the principle of deformation of path, since z_0, the point where the integrand in (5) is not analytic, is not a point of the annulus. This proves the formula for the a_n in (2).

(b) ***The negative powers*** in (1) and the formula for b_n in (2) are obtained if we consider $h(z)$. It consists of the second integral times $-1/(2\pi i)$ in (3). Since z lies in the annulus, it lies in the exterior of the path C_2. Hence the situation differs from that for the first integral. The essential point is that instead of [see (7*) in Sec. 15.4]

$$\text{(6)} \quad \text{(a)} \quad \left| \frac{z - z_0}{z^* - z_0} \right| < 1 \qquad \text{we now have} \qquad \text{(b)} \quad \left| \frac{z^* - z_0}{z - z_0} \right| < 1.$$

Consequently, we must develop the expression $1/(z^* - z)$ in the integrand of the second integral in (3) in powers of $(z^* - z_0)/(z - z_0)$ (instead of the reciprocal of this) to get a *convergent* series. We find

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \dfrac{z^* - z_0}{z - z_0}\right)}.$$

Compare this for a moment with (7) in Sec. 15.4, to really understand the difference. Then go on and apply formula (8), Sec. 15.4, for a finite geometric sum, obtaining

$$\frac{1}{z^* - z} = -\frac{1}{z - z_0}\left\{1 + \frac{z^* - z_0}{z - z_0} + \left(\frac{z^* - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{z^* - z_0}{z - z_0}\right)^n\right\}$$
$$- \frac{1}{z - z^*}\left(\frac{z^* - z_0}{z - z_0}\right)^{n+1}.$$

Multiplication by $-f(z^*)/2\pi i$ and integration over C_2 on both sides now yield

$$\begin{aligned} h(z) &= -\frac{1}{2\pi i}\oint_{C_2} \frac{f(z^*)}{z^* - z}\, dz^* \\ &= \frac{1}{2\pi i}\left\{\frac{1}{z - z_0}\oint_{C_2} f(z^*)\, dz^* + \frac{1}{(z - z_0)^2}\oint_{C_2} (z^* - z_0) f(z^*)\, dz^* + \cdots \right. \\ &\qquad + \frac{1}{(z - z_0)^n}\oint_{C_2} (z^* - z_0)^{n-1} f(z^*)\, dz^* \\ &\qquad \left. + \frac{1}{(z - z_0)^{n+1}}\oint_{C_2} (z^* - z_0)^n f(z^*)\, dz^*\right\} + R_n^*(z) \end{aligned}$$

with the last term on the right given by

$$R_n^*(z) = \frac{1}{2\pi i (z - z_0)^{n+1}}\oint_{C_2} \frac{(z^* - z_0)^{n+1}}{z - z^*} f(z^*)\, dz^*. \tag{7}$$

As before, we can integrate over C instead of C_2 in the integrals on the right. We see that on the right, the power $1/(z - z_0)^n$ is multiplied by b_n as given in (2). This establishes Laurent's theorem, provided

$$\lim_{n\to\infty} R_n^*(z) = 0. \tag{8}$$

(c) ***Convergence proof of* (8).** Very often (1) will have only finitely many negative powers. Then there is nothing to be proved. Otherwise, we begin by noting that $f(z^*)/(z - z^*)$ in (7) is bounded in absolute value, say,

$$\left|\frac{f(z^*)}{z - z^*}\right| < \widetilde{M} \qquad \text{for all } z^* \text{ on } C_2$$

because $f(z^*)$ is analytic in the annulus and on C_2, and z^* lies on C_2 and z outside, so that $z - z^* \neq 0$. From this and the ML-inequality (Sec. 14.1) applied to (7) we get the inequality ($L = 2\pi r_2$ = length of C_2, $r_2 = |z^* - z_0|$ = radius of C_2 = const)

$$|R_n^*(z)| \leqq \frac{1}{2\pi|z-z_0|^{n+1}} r_2^{n+1} \widetilde{M}L = \frac{\widetilde{M}L}{2\pi}\left(\frac{r_2}{|z-z_0|}\right)^{n+1}.$$

From (6b) we see that the expression on the right approaches zero as n approaches infinity. This proves (8). The representation (1) with coefficients (2) is now established in the given annulus.

(d) ***Convergence of*** **(1)** ***in the enlarged annulus.*** The first series in (1) is a Taylor series [representing $g(z)$]; hence it converges in the disk D with center z_0 whose radius equals the distance of the singularity (or singularities) closest to z_0. Also, $g(z)$ must be singular at all points outside C_1 where $f(z)$ is singular.

The second series in (1), representing $h(z)$, is a power series in $Z = 1/(z - z_0)$. Let the given annulus be $r_2 < |z - z_0| < r_1$, where r_1 and r_2 are the radii of C_1 and C_2, respectively (Fig. 370). This corresponds to $1/r_2 > |Z| > 1/r_1$. Hence this power series in Z must converge at least in the disk $|Z| < 1/r_2$. This corresponds to the exterior $|z - z_0| > r_2$ of C_2, so that $h(z)$ is analytic for all z outside C_2. Also, $h(z)$ must be singular inside C_2 where $f(z)$ is singular, and the series of the negative powers of (1) converges for all z in the exterior E of the circle with center z_0 and radius equal to the maximum distance from z_0 to the singularities of $f(z)$ inside C_2. The domain common to D and E is the enlarged open annulus characterized near the end of Laurent's theorem, whose proof is now complete. ■

Uniqueness. *The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique* (see Team Project 18). *However, $f(z)$ may have different Laurent series in two annuli with the same center*; see the examples below. The uniqueness is essential. As for a Taylor series, to obtain the coefficients of Laurent series, we do not generally use the integral formulas (2); instead, we use various other methods, some of which we shall illustrate in our examples. If a Laurent series has been found by any such process, the uniqueness guarantees that it must be ***the*** Laurent series of the given function in the given annulus.

EXAMPLE 1 Use of Maclaurin Series

Find the Laurent series of $z^{-5} \sin z$ with center 0.

Solution. By (14), Sec. 15.4, we obtain

$$z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040}z^2 + - \cdots \qquad (|z| > 0).$$

Here the "annulus" of convergence is the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6}z^{-2}$. ■

EXAMPLE 2 Substitution

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Solution. From (12) in Sec. 15.4 with z replaced by $1/z$ we obtain a Laurent series whose principal part is an infinite series,

$$z^2 e^{1/z} = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \qquad (|z| > 0). \quad ■$$

EXAMPLE 3 Development of 1/(1 − z)

Develop $1/(1-z)$ **(a)** in nonnegative powers of z, **(b)** in negative powers of z.

Solution.

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1).$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1).$$ ■

EXAMPLE 4 Laurent Expansions in Different Concentric Annuli

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

Solution. Multiplying by $1/z^3$, we get from Example 3

(I) $$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1),$$

(II) $$\frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$ ■

EXAMPLE 5 Use of Partial Fractions

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z+3}{z^2 - 3z + 2}$ with center 0.

Solution. In terms of partial fractions,

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

(a) and (b) in Example 3 take care of the first fraction. For the second fraction,

(c) $$-\frac{1}{z-2} = \frac{1}{2\left(1 - \frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

(d) $$-\frac{1}{z-2} = -\frac{1}{z\left(1 - \frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 371),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots.$$

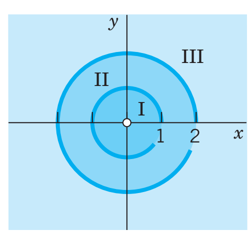


Fig. 371. Regions of convergence in Example 5

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots .$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots .$$ ■

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

PROBLEM SET 16.1

1–8 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$

2. $\dfrac{\exp(-1/z^2)}{z^2}$

3. $\dfrac{\exp z^2}{z^3}$

4. $\dfrac{\sin \pi z}{z^2}$

5. $\dfrac{1}{z^2 - z^3}$

6. $\dfrac{\sinh 2z}{z^2}$

7. $z^3 \cosh \dfrac{1}{z}$

8. $\dfrac{e^z}{z^2 - z^3}$

9–16 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

9. $\dfrac{e^z}{(z-1)^2}, \quad z_0 = 1$

10. $\dfrac{z^2 - 3i}{(z-3)^2}, \quad z_0 = 3$

11. $\dfrac{z^2}{(z - \pi i)^4}, \quad z_0 = \pi i$

12. $\dfrac{1}{z^2(z-i)}, \quad z_0 = i$

13. $\dfrac{1}{z^3(z-i)^2}, \quad z_0 = i$

14. $\dfrac{e^{az}}{z-b}, \quad z_0 = b$

15. $\dfrac{\cos z}{(z-\pi)^2}, \quad z_0 = \pi$

16. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}, \quad z_0 = \frac{1}{4}\pi$

17. **CAS PROJECT. Partial Fractions.** Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.

18. **TEAM PROJECT. Laurent Series. (a) Uniqueness.** Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

    (b) Accumulation of singularities. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

    (c) Integrals. Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2} \int_0^z \frac{e^t - 1}{t}\, dt, \qquad \frac{1}{z^3} \int_0^z \frac{\sin t}{t}\, dt.$$

19–25 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

19. $\dfrac{1}{1 - z^2}, \quad z_0 = 0$

20. $\dfrac{1}{z}, \quad z_0 = 1$

21. $\dfrac{\sin z}{z + \frac{1}{2}\pi}, \quad z_0 = -\frac{1}{2}\pi$

22. $\dfrac{1}{z^2}, \quad z_0 = i$

23. $\dfrac{z^8}{1 - z^4}, \quad z_0 = 0$

24. $\dfrac{\sinh z}{(z-1)^4}, \quad z_0 = 1$

25. $\dfrac{z^3 - 2iz^2}{(z-i)^2}, \quad z_0 = i$

16.2 Singularities and Zeros. Infinity

Roughly, a *singular point* of an analytic function $f(z)$ is a z_0 at which $f(z)$ ceases to be analytic, and a *zero* is a z at which $f(z) = 0$. Precise definitions follow below. In this section we show that Laurent series can be used for classifying singularities and Taylor series for discussing zeros.

Singularities were defined in Sec. 15.4, as we shall now recall and extend. We also remember that, by definition, a function is a ***single-valued*** relation, as was emphasized in Sec. 13.3.

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* tan z has isolated singularities at $\pm\pi/2$, $\pm 3\pi/2$, etc.; tan $(1/z)$ has a nonisolated singularity at 0. (Explain!)

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1) \quad \text{(Sec. 16.1)}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called the **principal part** of (1), as we remember from the last section. If it has only finitely many terms, it is of the form

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0). \qquad (2)$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole**, and m is called its **order**. Poles of the first order are also known as **simple poles**.

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity**.

We leave aside nonisolated singularities.

EXAMPLE 1 Poles. Essential Singularities

The function

$$f(z) = \frac{1}{z(z - 2)^5} + \frac{3}{(z - 2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots .$$

Section 16.1 provides further examples. In that section, Example 1 shows that $z^{-5}\sin z$ has a fourth-order pole at 0. Furthermore, Example 4 shows that $1/(z^3 - z^4)$ has a third-order pole at 0 and a Laurent series with infinitely many negative powers. This is no contradiction, since this series is valid for $|z| > 1$; it merely tells us that in classifying singularities it is quite important to consider the Laurent series valid ***in the immediate neighborhood*** of a singular point. In Example 4 this is the series (I), which has three negative powers. ■

The classification of singularities into poles and essential singularities is not merely a formal matter, because the behavior of an analytic function in a neighborhood of an essential singularity is entirely different from that in the neighborhood of a pole.

EXAMPLE 2 Behavior Near a Pole

$f(z) = 1/z^2$ has a pole at $z = 0$, and $|f(z)| \to \infty$ as $z \to 0$ in any manner. This illustrates the following theorem. ■

THEOREM 1 Poles

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

The proof is left as an exercise (see Prob. 24).

EXAMPLE 3 Behavior Near an Essential Singularity

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. It has no limit for approach along the imaginary axis; it becomes infinite if $z \to 0$ through positive real values, but it approaches zero if $z \to 0$ through negative real values. It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small ϵ-neighborhood of $z = 0$. To see the latter, we set $z = re^{i\theta}$, and then obtain the following complex equation for r and θ, which we must solve:

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}.$$

Equating the absolute values and the arguments, we have $e^{(\cos\theta)/r} = c_0$, that is

$$\cos\theta = r\ln c_0, \qquad \text{and} \qquad -\sin\theta = \alpha r$$

respectively. From these two equations and $\cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$ we obtain the formulas

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \qquad \text{and} \qquad \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Hence r can be made arbitrarily small by adding multiples of 2π to α, leaving c unaltered. This illustrates the very famous *Picard's theorem* (with $z = 0$ as the exceptional value). ■

THEOREM 2 Picard's Theorem

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ϵ-neighborhood of z_0.

For the rather complicated proof, see Ref. [D4], vol. 2, p. 258. For historical information on Picard, see footnote 9 in Problem Set 1.7.

Removable Singularities. We say that a function $f(z)$ has a *removable singularity* at $z = z_0$ if $f(z)$ is not analytic at $z = z_0$, but can be made analytic there by assigning a suitable value $f(z_0)$. Such singularities are of no interest since they can be removed as just indicated. *Example:* $f(z) = (\sin z)/z$ becomes analytic at $z = 0$ if we define $f(0) = 1$.

Zeros of Analytic Functions

A **zero** of an analytic function $f(z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$. A zero has **order** n if not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)}(z_0) \neq 0$. A first-order zero is also called a **simple zero**. For a second-order zero, $f(z_0) = f'(z_0) = 0$ but $f''(z_0) \neq 0$. And so on.

EXAMPLE 4 Zeros

The function $1 + z^2$ has simple zeros at $\pm i$. The function $(1 - z^4)^2$ has second-order zeros at ± 1 and $\pm i$. The function $(z - a)^3$ has a third-order zero at $z = a$. The function e^z has no zeros (see Sec. 13 5). The function $\sin z$ has simple zeros at $0, \pm\pi, \pm 2\pi, \cdots$, and $\sin^2 z$ has second-order zeros at these points. The function $1 - \cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \cdots$, and the function $(1 - \cos z)^2$ has fourth-order zeros at these points. ■

Taylor Series at a Zero. At an nth-order zero $z = z_0$ of $f(z)$, the derivatives $f'(z_0), \cdots, f^{(n-1)}(z_0)$ are zero, by definition. Hence the first few coefficients $a_0, \cdots, a_{n-1}$ of the Taylor series (1), Sec. 15.4, are zero, too, whereas $a_n \neq 0$, so that this series takes the form

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\ &= (z - z_0)^n \left[a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots\right] \qquad (a_n \neq 0). \end{aligned} \tag{3}$$

This is characteristic of such a zero, because, if $f(z)$ has such a Taylor series, it has an nth-order zero at $z = z_0$, as follows by differentiation.

Whereas nonisolated singularities may occur, for zeros we have

THEOREM 3 Zeros

The zeros of an analytic function $f(z)$ $(\not\equiv 0)$ are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

PROOF The factor $(z - z_0)^n$ in (3) is zero only at $z = z_0$. The power series in the brackets $[\cdots]$ represents an analytic function (by Theorem 5 in Sec. 15.3), call it $g(z)$. Now $g(z_0) = a_n \neq 0$, since an analytic function is continuous, and because of this continuity, also $g(z) \neq 0$ in some neighborhood of $z = z_0$. Hence the same holds of $f(z)$. ■

This theorem is illustrated by the functions in Example 4.

Poles are often caused by zeros in the denominator. (*Example:* $\tan z$ has poles where $\cos z$ is zero.) This is a major reason for the importance of zeros. The key to the connection is the following theorem, whose proof follows from (3) (see Team Project 12).

THEOREM 4 Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of nth order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

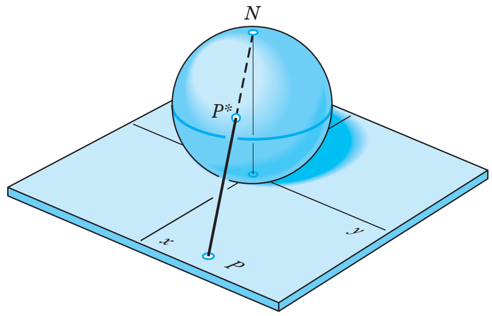


Fig. 372. Riemann sphere

Riemann Sphere. Point at Infinity

When we want to study complex functions for large $|z|$, the complex plane will generally become rather inconvenient. Then it may be better to use a representation of complex numbers on the so-called **Riemann sphere**. This is a sphere S of diameter 1 touching the complex z-plane at $z = 0$ (Fig. 372), and we let the image of a point P (a number z in the plane) be the intersection P^* of the segment PN with S, where N is the "North Pole" diametrically opposite to the origin in the plane. Then to each z there corresponds a point on S.

Conversely, each point on S represents a complex number z, except for N, which does not correspond to any point in the complex plane. This suggests that we introduce an additional point, called the **point at infinity** and denoted ∞ ("infinity") and let its image be N. The complex plane together with ∞ is called the **extended complex plane**. The complex plane is often called the *finite complex plane*, for distinction, or simply the *complex plane* as before. The sphere S is called the **Riemann sphere**. The mapping of the extended complex plane onto the sphere is known as a **stereographic projection**. (What is the image of the Northern Hemisphere? Of the Western Hemisphere? Of a straight line through the origin?)

Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic** or **singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

$$g(0) = \lim_{w \to 0} g(w) \tag{4}$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

EXAMPLE 5 Functions Analytic or Singular at Infinity. Entire and Meromorphic Functions

The function $f(z) = 1/z^2$ is analytic at ∞ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$, and $f(z)$ has a second-order zero at ∞. The function $f(z) = z^3$ is singular at ∞ and has a third-order pole there since the function $g(w) = f(1/w) = 1/w^3$ has such a pole at $w = 0$. The function e^z has an essential singularity at ∞ since $e^{1/w}$ has such a singularity at $w = 0$. Similarly, $\cos z$ and $\sin z$ have an essential singularity at ∞.

Recall that an **entire function** is one that is analytic everywhere in the (finite) complex plane. Liouville's theorem (Sec. 14.4) tells us that the only *bounded* entire functions are the constants, hence any nonconstant entire function must be unbounded. Hence it has a singularity at ∞, a pole if it is a polynomial or an essential singularity if it is not. The functions just considered are typical in this respect.

An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function.** Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ■

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

PROBLEM SET 16.2

1–10 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z^4 - 81)^3$
3. $(z + 81i)^4$
4. $\tan^2 2z$
5. $z^{-2} \sin^2 \pi z$
6. $\cosh^4 z$
7. $z^4 + (1 - 8i)z^2 - 8i$
8. $(\sin z - 1)^3$
9. $\sin 2z \cos 2z$
10. $(z^2 - 8)^3(\exp(z^2) - 1)$

11. **Zeros.** If $f(z)$ is analytic and has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$ at z_0.

12. **TEAM PROJECT. Zeros. (a) Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at z_0.

    (b) Poles and zeros. Prove Theorem 4.

    (c) Isolated *k*-points. Show that the points at which a nonconstant analytic function $f(z)$ has a given value k are isolated.

    (d) Identical functions. If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.

13–22 SINGULARITIES

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

13. $\dfrac{1}{(z + 2i)^2} - \dfrac{z}{z - i} + \dfrac{z + 1}{(z - i)^2}$

14. $e^{z-i} + \dfrac{2}{z - i} - \dfrac{8}{(z - i)^3}$

15. $z \exp(1/(z - 1 - i)^2)$
16. $\tan \pi z$
17. $\cot^4 z$
18. $z^3 \exp\left(\dfrac{1}{z - 1}\right)$
19. $1/(e^z - e^{2z})$
20. $1/(\cos z - \sin z)$
21. $e^{1/(z-1)}/(e^z - 1)$
22. $(z - \pi)^{-1} \sin z$

23. **Essential singularity.** Discuss e^{1/z^2} in a similar way as $e^{1/z}$ is discussed in Example 3 of the text.

24. **Poles.** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

25. **Riemann sphere.** Assuming that we let the image of the x-axis be the meridians 0° and 180°, describe and sketch (or graph) the images of the following regions on the Riemann sphere: **(a)** $|z| > 100$, **(b)** the lower half-plane, **(c)** $\frac{1}{2} \leqq |z| \leqq 2$.

16.3 Residue Integration Method

We now cover a second method of evaluating complex integrals. Recall that we solved complex integrals directly by Cauchy's integral formula in Sec. 14.3. In Chapter 15 we learned about power series and especially Taylor series. We generalized Taylor series to Laurent series (Sec. 16.1) and investigated singularities and zeroes of various functions (Sec. 16.2). Our hard work has paid off and we see how much of the theoretical groundwork comes together in evaluating complex integrals by the residue method.

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\, dz$$

taken around a simple closed path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

The situation changes if $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$ (sometimes called a **deleted neighborhood**, an old-fashioned term that we shall not use). Now comes the key idea. The coefficient b_1 of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) in Sec. 16.1 with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Now, since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients (see the examples in Sec. 16.1), we can find b_1 by one of those methods and then use the formula for b_1 for evaluating the integral, that is,

$$\oint_C f(z)\, dz = 2\pi i b_1. \tag{1}$$

Here we integrate counterclockwise around a simple closed path C that contains $z = z_0$ in its interior (but no other singular points of $f(z)$ on or inside C!).

The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we denote it by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z). \tag{2}$$

EXAMPLE 1 **Evaluation of an Integral by Means of a Residue**

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Solution. From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -\frac{1}{3}!$. From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

EXAMPLE 2 **CAUTION! Use the Right Laurent Series!**

Integrate $f(z) = 1/(z^3 - z^4)$ clockwise around the circle C: $|z| = \frac{1}{2}$.

Solution. $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$. Now $z = 1$ lies outside C. Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1).$$

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

CAUTION! Had we used the wrong series (II) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots \qquad (|z| > 1),$$

we would have obtained the wrong answer, 0, because this series has no power $1/z$. ■

Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

Simple Poles at z_0. A first formula for the residue at a simple pole is

(3) $$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z). \qquad \text{(Proof below).}$$

A second formula for the residue at a simple pole is

(4) $$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \qquad \text{(Proof below).}$$

In (4) we assume that $f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z)$ has a simple pole at z_0 by Theorem 4 in Sec. 16.2.

PROOF We prove (3). For a simple pole at $z = z_0$ the Laurent series (1), Sec. 16.1, is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R).$$

Here $b_1 \neq 0$. (Why?) Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain the formula (3):

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots] = b_1$$

where the last equality follows from continuity (Theorem 1, Sec. 15.3).

We prove (4). The Taylor series of $q(z)$ at a simple zero z_0 is

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots.$$

Substituting this into $f = p/q$ and then f into (3) gives

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''(z_0)/2 + \cdots]}.$$

$z - z_0$ cancels. By continuity, the limit of the denominator is $q'(z_0)$ and (4) follows. ■

EXAMPLE 3 **Residue at a Simple Pole**

$f(z) = (9z + i)/(z^3 + z)$ has a simple pole at i because $z^2 + 1 = (z + i)(z - i)$, and (3) gives the residue

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z\to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

By (4) with $p(i) = 9i + i$ and $q'(z) = 3z^2 + 1$ we confirm the result,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$ ■

Poles of Any Order at z_0. The residue of $f(z)$ at an mth-order pole at z_0 is

(5)
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m - 1)!} \lim_{z\to z_0} \left\{\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]\right\}.$$

In particular, for a second-order pole ($m = 2$),

(5*)
$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \{[(z - z_0)^2 f(z)]'\}.$$

PROOF We prove (5). The Laurent series of $f(z)$ converging near z_0 (except at z_0 itself) is (Sec. 16.2)

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

where $b_m \neq 0$. The residue wanted is b_1. Multiplying both sides by $(z - z_0)^m$ gives

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots.$$

We see that b_1 is now the coefficient of the power $(z - z_0)^{m-1}$of the power series of $g(z) = (z - z_0)^m f(z)$. Hence Taylor's theorem (Sec. 15.4) gives (5):

$$b_1 = \frac{1}{(m - 1)!} g^{(m-1)}(z_0)$$
$$= \frac{1}{(m - 1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$ ■

EXAMPLE 4 **Residue at a Pole of Higher Order**

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$ has a pole of second order at $z = 1$ because the denominator equals $(z + 4)(z - 1)^2$ (verify!). From (5*) we obtain the residue

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz} [(z - 1)^2 f(z)] = \lim_{z\to 1} \frac{d}{dz} \left(\frac{50z}{z + 4}\right) = \frac{200}{5^2} = 8.$$ ■

Several Singularities Inside the Contour. Residue Theorem

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C. This is the purpose of the residue theorem. The extension is surprisingly simple.

THEOREM 1

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:

(6)
$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z).$$

PROOF We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated (Fig. 373 where $k = 3$). Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

(7)
$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz = 0,$$

the integral along C being taken *counterclockwise* and the other integrals *clockwise* (as in Figs. 354 and 355, Sec. 14.2). We take the integrals over $C_1, \cdots, C_k$ to the right and compensate the resulting minus sign by reversing the sense of integration. Thus,

(8)
$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz$$

where all the integrals are now taken counterclockwise. By (1) and (2),

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z), \qquad j = 1, \cdots, k,$$

so that (8) gives (6) and the residue theorem is proved. ■

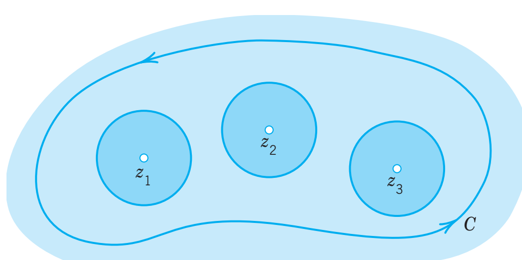


Fig. 373. Residue theorem

This important theorem has various applications in connection with complex and real integrals. Let us first consider some complex integrals. (Real integrals follow in the next section.)

EXAMPLE 5 Integration by the Residue Theorem. Several Contours

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4).] *Answer:* (a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ■

EXAMPLE 6 Another Application of the Residue Theorem

Integrate $(\tan z)/(z^2-1)$ counterclockwise around the circle C: $|z| = \frac{3}{2}$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2-1 = (z-1)(z+1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\oint_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1} \frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1} \frac{\tan z}{z^2-1}\right)$$

$$= 2\pi i\left(\frac{\tan z}{2z}\bigg|_{z=1} + \frac{\tan z}{2z}\bigg|_{z=-1}\right)$$

$$= 2\pi i \tan 1 = 9.7855i.$$ ■

EXAMPLE 7 Poles and Essential Singularities

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz.$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots \qquad (|z| > 0).$$

Answer: $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem. ■

PROBLEM SET 16.3

1. Verify the calculations in Example 3 and find the other residues.
2. Verify the calculations in Example 4 and find the other residue.

3–12 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

3. $\dfrac{\sin 2z}{z^6}$

4. $\dfrac{\cos z}{z^4}$

5. $\dfrac{8}{1 + z^2}$

6. $\tan z$

7. $\cot \pi z$

8. $\dfrac{\pi}{(z^2 - 1)^2}$

9. $\dfrac{1}{1 - e^z}$

10. $\dfrac{z^4}{z^2 - iz + 2}$

11. $\dfrac{e^z}{(z - \pi i)^3}$

12. $e^{1/(1-z)}$

13. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order in the finite plane. Use it for solving Probs. 5–10.

14–25 RESIDUE INTEGRATION

Evaluate (counterclockwise). Show the details.

14. $\displaystyle\oint_C \frac{z - 23}{z^2 - 4z - 5}\, dz, \quad C\colon |z - 2 - i| = 3.2$

15. $\displaystyle\oint_C \tan 2\pi z\, dz, \quad C\colon |z - 0.2| = 0.2$

16. $\displaystyle\oint_C e^{1/z}\, dz, \quad C\text{: the unit circle}$

17. $\displaystyle\oint_C \frac{e^z}{\cos z}\, dz, \quad C\colon |z - \pi i/2| = 4.5$

18. $\displaystyle\oint_C \frac{z + 1}{z^4 - 2z^3}\, dz, \quad C\colon |z - 1| = 2$

19. $\displaystyle\oint_C \frac{\sinh z}{2z - i}\, dz, \quad C\colon |z - 2i| = 2$

20. $\displaystyle\oint_C \frac{dz}{(z^2 + 1)^3}, \quad C\colon |z - i| = 3$

21. $\displaystyle\oint_C \frac{\cos \pi z}{z^5}\, dz, \quad C\colon |z| = \tfrac{1}{2}$

22. $\displaystyle\oint_C \frac{z^2 \sin z}{4z^2 - 1}\, dz, \quad C \text{ the unit circle}$

23. $\displaystyle\oint_C \frac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)}\, dz, \quad C \text{ the unit circle}$

24. $\displaystyle\oint_C \frac{\exp(-z^2)}{\sin 4z}\, dz, \quad C\colon |z| = 1.5$

25. $\displaystyle\oint_C \frac{z \cosh \pi z}{z^4 + 13z^2 + 36}\, dz, \quad |z| = \pi$

16.4 Residue Integration of Real Integrals

Surprisingly, residue integration can also be used to evaluate certain classes of complicated real integrals. This shows an advantage of complex analysis over real analysis or calculus.

Integrals of Rational Functions of cos θ and sin θ

We first consider integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\textbf{(2)}\qquad \begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right). \end{aligned}$$

Since F is rational in $\cos\theta$ and $\sin\theta$, Eq. (2) shows that F is now a rational function of z, say, $f(z)$. Since $dz/d\theta = ie^{i\theta}$, we have $d\theta = dz/iz$ and the given integral takes the form

$$(3)\qquad J = \oint_C f(z)\,\frac{dz}{iz}$$

and, as θ ranges from 0 to 2π in (1), the variable $z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$. (Review Sec. 13.5 if necessary.)

EXAMPLE 1 An Integral of the Type (1)

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

Solution. We use $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = dz/iz$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

We see that the integrand has a simple pole at $z_1 = \sqrt{2} + 1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2} - 1$ (where $z - \sqrt{2} + 1 = 0$) inside C with residue [by (3), Sec. 16.3]

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$

$$= -\frac{1}{2}.$$

Answer: $2\pi i(-2/i)(-\frac{1}{2}) = 2\pi$. (Here $-2/i$ is the factor in front of the last integral.) ■

As another large class, let us consider real integrals of the form

$$(4)\qquad \int_{-\infty}^{\infty} f(x)\,dx.$$

Such an integral, whose interval of integration is not finite is called an **improper integral**, and it has the meaning

$$(5')\qquad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx.$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{5}$$

The limit in (5) is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx.$$

It may exist even if the limits in (5′) do not. *Example:*

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2}\right) = 0, \qquad \text{but} \qquad \lim_{b\to\infty} \int_{0}^{b} x\,dx = \infty.$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of the numerator. Then the limits in (5′) exist, and we may start from (5). We consider the corresponding contour integral

$$\oint_C f(z)\,dz \tag{5*}$$

around a path C in Fig. 374. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

where the sum consists of all the residues of $f(z)$ at the points in the upper half-plane at which $f(z)$ has a pole. From this we have

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)\,dz. \tag{6}$$

We prove that, if $R \to \infty$, the value of the integral over the semicircle S approaches zero. If we set $z = Re^{i\theta}$, then S is represented by $R = \text{const}$, and as z ranges along S, the variable θ ranges from 0 to π. Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

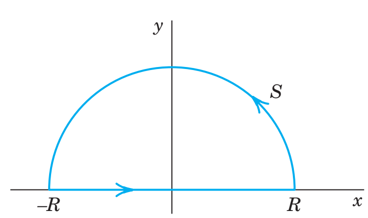


Fig. 374. Path C of the contour integral in (5*)

for sufficiently large constants k and R_0. By the ML-inequality in Sec. 14.1,

$$\left|\int_S f(z)\,dz\right| < \frac{k}{R^2}\,\pi R = \frac{k\pi}{R} \qquad (R > R_0).$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) \tag{7}$$

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

EXAMPLE 2 An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

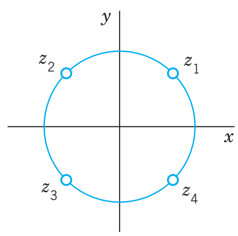


Fig. 375. Example 2

Solution. Indeed, $f(z) = 1/(1+z^4)$ has four simple poles at the points (make a sketch)

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of these poles lie in the upper half-plane (Fig. 375). From (4) in the last section we find the residues

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}.$$

(Here we used $e^{\pi i} = -1$ and $e^{-2\pi i} = 1$.) By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = -\frac{2\pi i}{4}\left(e^{\pi i/4} - e^{-\pi i/4}\right) = -\frac{2\pi i}{4}\cdot 2i\sin\frac{\pi}{4} = \pi\sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1 + x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$ ■

Fourier Integrals

The method of evaluating (4) by creating a closed contour (Fig. 374) and "blowing it up" extends to integrals

$$\text{(8)} \qquad \int_{-\infty}^{\infty} f(x) \cos sx \, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x) \sin sx \, dx \qquad (s \text{ real})$$

as they occur in connection with the Fourier integral (Sec. 11.7).

If $f(x)$ is a rational function satisfying the assumption on the degree as for (4), we may consider the corresponding integral

$$\oint_C f(z) e^{isz} \, dz \qquad (s \text{ real and positive})$$

over the contour C in Fig. 374. Instead of (7) we now get

$$\text{(9)} \qquad \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = 2\pi i \sum \text{Res} \, [f(z) e^{isz}] \qquad (s > 0)$$

where we sum the residues of $f(z)e^{isz}$ at its poles in the upper half-plane. Equating the real and the imaginary parts on both sides of (9), we have

$$\textbf{(10)} \qquad \begin{aligned} \int_{-\infty}^{\infty} f(x) \cos sx \, dx &= -2\pi \sum \text{Im Res} \, [f(z) e^{isz}], \\ \int_{-\infty}^{\infty} f(x) \sin sx \, dx &= 2\pi \sum \text{Re Res} \, [f(z) e^{isz}]. \end{aligned} \qquad (s > 0)$$

To establish (9), we must show [as for (4)] that the value of the integral over the semicircle S in Fig. 374 approaches 0 as $R \to \infty$. Now $s > 0$ and S lies in the upper half-plane $y \geqq 0$. Hence

$$|e^{isz}| = |e^{is(x+iy)}| = |e^{isx}||e^{-sy}| = 1 \cdot e^{-sy} \leqq 1 \qquad (s > 0, \quad y \geqq 0).$$

From this we obtain the inequality $|f(z)e^{isz}| = |f(z)||e^{isz}| \leqq |f(z)| \quad (s > 0, \quad y \geqq 0)$. This reduces our present problem to that for (4). Continuing as before gives (9) and (10). ■

EXAMPLE 3 An Application of (10)

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2} \, dx = \frac{\pi}{k} e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2} \, dx = 0 \qquad (s > 0, k > 0).$$

Solution. In fact, $e^{isz}/(k^2 + z^2)$ has only one pole in the upper half-plane, namely, a simple pole at $z = ik$, and from (4) in Sec. 16.3 we obtain

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\, dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{-ks}.$$

Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results [see also (15) in Sec. 11.7.]

Another Kind of Improper Integral

We consider an improper integral

(11) $$\int_A^B f(x)\, dx$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty.$$

By definition, this integral (11) means

(12) $$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{a+\eta}^B f(x)\, dx$$

where both ϵ and η approach zero independently and through positive values. It may happen that neither of these two limits exists if ϵ and η go to 0 independently, but the limit

(13) $$\lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\, dx + \int_{a+\epsilon}^B f(x)\, dx\right]$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_A^B f(x)\, dx.$$

For example,

$$\text{pr. v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3}\right] = 0;$$

the principal value exists, although the integral itself has no meaning.

In the case of simple poles on the real axis we shall obtain a formula for the principal value of an integral from $-\infty$ to ∞. This formula will result from the following theorem.

THEOREM 1

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 376)

$$\lim_{r \to 0} \int_{C_2} f(z)\, dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

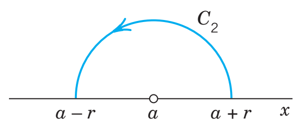


Fig. 376. Theorem 1

PROOF By the definition of a simple pole (Sec. 16.2) the integrand $f(z)$ has for $0 < |z - a| < R$ the Laurent series

$$f(z) = \frac{b_1}{z - a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z).$$

Here $g(z)$ is analytic on the semicircle of integration (Fig. 376)

$$C_2: \quad z = a + re^{i\theta}, \qquad 0 \leqq \theta \leqq \pi$$

and for all z between C_2 and the x-axis, and thus bounded on C_2, say, $|g(z)| \leqq M$. By integration,

$$\int_{C_2} f(z)\, dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} ire^{i\theta}\, d\theta + \int_{C_2} g(z)\, dz = b_1 \pi i + \int_{C_2} g(z)\, dz.$$

The second integral on the right cannot exceed $M\pi r$ in absolute value, by the ML-inequality (Sec. 14.1), and $ML = M\pi r \to 0$ as $r \to 0$. ■

Figure 377 shows the idea of applying Theorem 1 to obtain the principal value of the integral of a rational function $f(x)$ from $-\infty$ to ∞. For sufficiently large R the integral over the entire contour in Fig. 377 has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane. We assume that $f(x)$ satisfies the degree condition imposed in connection with (4). Then the value of the integral over the large

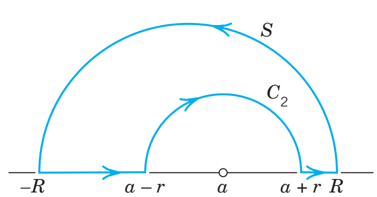


Fig. 377. Application of Theorem 1

semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

(14)
$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2 - 3z + 2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3-i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}.$$ ■

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

PROBLEM SET 16.4

1–9 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of your work.

1. $\displaystyle\int_0^{\pi} \frac{2\,d\theta}{k - \cos\theta}$

2. $\displaystyle\int_0^{\pi} \frac{d\theta}{\pi + 3\cos\theta}$

3. $\displaystyle\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\,d\theta$

4. $\displaystyle\int_0^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta}\,d\theta$

5. $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\cos\theta}\,d\theta$

6. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$

7. $\displaystyle\int_0^{2\pi} \frac{a}{a - \sin\theta}\,d\theta$

8. $\displaystyle\int_0^{2\pi} \frac{1}{8 - 2\sin\theta}\,d\theta$

9. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\,d\theta$

10–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$

12. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2 + 1)(x^2 + 4)}\,dx$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\,dx$

15. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1}\,dx$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2 + 1)^2}\,dx$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{x^4 + 1}\,dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4 + 5x^2 + 4}\,dx$

19. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

20. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{8 - x^3}\,dx$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{(x - 1)(x^2 + 4)}\,dx$

22. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - ix}$

23–26 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

23. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

24. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$

25. $\displaystyle\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\,dx$

27. **CAS EXPERIMENT. Simple Poles on the Real Axis.** Experiment with integrals $\int_{-\infty}^{\infty} f(x)\,dx$, $f(x) = [(x - a_1)(x - a_2)\cdots(x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.

28. **TEAM PROJECT. Comments on Real Integrals.**
(a) Formula (10) follows from (9). Give the details.
(b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a, a, a + ib, -a + ib$, letting $a \to \infty$, and using

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^{\infty} e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

(This integral is needed in heat conduction in Sec. 12.7.)
(c) Inspection. Solve Probs. 13 and 17 without calculation.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. What is a Laurent series? Its principal part? Its use? Give simple examples.
2. What kind of singularities did we discuss? Give definitions and examples.
3. What is the residue? Its role in integration? Explain methods to obtain it.
4. Can the residue at a singularity be zero? At a simple pole? Give reason.
5. State the residue theorem and the idea of its proof from memory.
6. How did we evaluate real integrals by residue integration? How did we obtain the closed paths needed?

7. What are improper integrals? Their principal value? Why did they occur in this chapter?

8. What do you know about zeros of analytic functions? Give examples.

9. What is the extended complex plane? The Riemann sphere R? Sketch $z = 1 + i$ on R.

10. What is an entire function? Can it be analytic at infinity? Explain the definitions.

11–18 COMPLEX INTEGRALS

Integrate counterclockwise around C. Show the details.

11. $\frac{\sin 3z}{z^2}$, $C: |z| = \pi$

12. $e^{2/z}$, $C: |z - 1 - i| = 2$

13. $\frac{5z^3}{z^2 + 4}$, $C: |z| = 3$

14. $\frac{5z^3}{z^2 + 4}$, $C: |z - i| = \pi i/2$

15. $\frac{25z^2}{(z - 5)^2}$, $C: |z - 5| = 1$

16. $\frac{15z + 9}{z^3 - 9z}$, $C: |z| = 4$

17. $\frac{\cos z}{z^n}$, $n = 0, 1, 2, \cdots$, $C: |z| = 1$

18. $\cot 4z$, $C: |z| = \frac{3}{4}$

19–25 REAL INTEGRALS

Evaluate by the methods of this chapter. Show details.

19. $\int_0^{2\pi} \frac{d\theta}{13 - 5 \sin \theta}$

20. $\int_0^{2\pi} \frac{\sin \theta}{3 + \cos \theta}\, d\theta$

21. $\int_0^{2\pi} \frac{\sin \theta}{34 - 16 \sin \theta}\, d\theta$

22. $\int_{-\infty}^{\infty} \frac{dx}{1 + 4x^4}$

23. $\int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2}\, dx$

24. $\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4ix}$

25. $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1}\, dx$

SUMMARY OF CHAPTER 16
Laurent Series. Residue Integration

A **Laurent series** is a series of the form

(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ (Sec. 16.1)

or, more briefly written [but this means the same as (1)!]

(1*) $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*$$

where $n = 0, \pm 1, \pm 2, \cdots$. This series converges in an open annulus (ring) A with center z_0. In A the function $f(z)$ is analytic. At points not in A it may have singularities. The first series in (1) is a power series. In a given annulus, a Laurent series of $f(z)$ is unique, but $f(z)$ may have different Laurent series in different annuli with the same center.

Of particular importance is the Laurent series (1) that converges in a neighborhood of z_0 except at z_0 itself, say, for $0 < |z - z_0| < R$ ($R > 0$, suitable). The series (or finite sum) of the negative powers in this Laurent series is called the **principal part** of $f(z)$ at z_0. The coefficient b_1 of $1/(z - z_0)$ in this series is called the **residue** of $f(z)$ at z_0 and is given by [see (1) and (1*)]

$$(2) \quad b_1 = \operatorname*{Res}_{z \to z_0} f(z) = \frac{1}{2\pi i} \oint_C f(z^*)\, dz^*. \qquad \text{Thus} \qquad \oint_C f(z^*)\, dz^* = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

b_1 can be used for ***integration*** as shown in (2) because it can be found from

$$(3) \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right), \qquad \text{(Sec. 16.3)},$$

provided $f(z)$ has at z_0 a **pole of order *m***; by definition this means that principal part has $1/(z - z_0)^m$ as its highest negative power. Thus for a simple pole ($m = 1$),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z); \qquad \text{also,} \qquad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

If the principal part is an infinite series, the singularity of $f(z)$ at z_0 is called an **essential singularity** (Sec. 16.2).

Section 16.2 also discusses the *extended complex plane*, that is, the complex plane with an improper point ∞ ("infinity") attached.

Residue integration may also be used to evaluate certain classes of complicated real integrals (Sec. 16.4).

CHAPTER 17

Conformal Mapping

Conformal mappings are invaluable to the engineer and physicist as an aid in solving problems in potential theory. They are a standard method for solving *boundary value problems* in two-dimensional potential theory and yield rich applications in electrostatics, heat flow, and fluid flow, as we shall see in Chapter 18.

The main feature of conformal mappings is that they are angle-preserving (except at some critical points) and allow a *geometric approach to complex analysis.* More details are as follows. Consider a complex function $w = f(z)$ defined in a domain D of the z–plane; then to each point in D there corresponds a point in the w-plane. In this way we obtain a **mapping** of D onto the range of values of $f(z)$ in the w-plane. In Sec. 17.1 we show that if $f(z)$ is an analytic function, then the mapping given by $w = f(z)$ is a **conformal mapping**, that is, it preserves angles, except at points where the derivative $f'(z)$ is zero. (Such points are called critical points.)

Conformality appeared early in the history of construction of maps of the globe. Such maps can be either "conformal," that is, give directions correctly, or "equiareal," that is, give areas correctly except for a scale factor. However, the maps will always be distorted because they cannot have both properties, as can be proven, see [GenRef8] in App. 1. The designer of accurate maps then has to select which distortion to take into account.

Our study of conformality is similar to the approach used in calculus where we study properties of real functions $y = f(x)$ and graph them. Here we study the properties of conformal mappings (Secs. 17.1–17.4) to get a deeper understanding of the properties of functions, most notably the ones discussed in Chap. 13. Chapter 17 ends with an introduction to ***Riemann surfaces***, an ingenious geometric way of dealing with multivalued complex functions such as $w = \text{sqrt}\,(z)$ and $w = \ln z$.

So far we have covered two main approaches to solving problems in complex analysis. The first one was solving complex integrals by Cauchy's integral formula and was broadly covered by material in Chaps. 13 and 14. The second approach was to use Laurent series and solve complex integrals by residue integration in Chaps. 15 and 16. Now, in Chaps. 17 and 18, we develop a third approach, that is, the *geometric approach* of conformal mapping to solve boundary value problems in complex analysis.

Prerequisite: Chap. 13.
Sections that may be omitted in a shorter course: 17.3 and 17.5.
References and Answers to Problems: App. 1 Part D, App. 2.

17.1 Geometry of Analytic Functions: Conformal Mapping

We shall see that conformal mappings are those mappings that preserve angles, except at critical points, and that these mappings are defined by analytic functions. A critical point occurs wherever the derivative of such a function is zero. To arrive at these results, we have to define terms more precisely.

A complex function

$$w = f(z) = u(x, y) + iv(x, y) \qquad (z = x + iy) \tag{1}$$

of a complex variable z gives a **mapping** of its domain of definition D in the complex z-plane ***into*** the complex w-plane or ***onto*** its range of values in that plane.[1] For any point z_0 in D the point $w_0 = f(z_0)$ is called the **image** of z_0 with respect to f. More generally, for the points of a curve C in D the image points form the **image** of C; similarly for other point sets in D. Also, instead of *the mapping by a function* $w = f(z)$ we shall say more briefly ***the mapping*** $\boldsymbol{w = f(z)}$.

EXAMPLE 1 Mapping $w = f(x) = z^2$

Using polar forms $z = re^{i\theta}$ and $w = Re^{i\phi}$, we have $w = z^2 = r^2e^{2i\theta}$. Comparing moduli and arguments gives $R = r^2$ and $\phi = 2\theta$. Hence circles $r = r_0$ are mapped onto circles $R = r_0^2$ and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$. Figure 378 shows this for the region $1 \leqq |z| \leqq \frac{3}{2}$, $\pi/6 \leqq \theta \leqq \pi/3$, which is mapped onto the region $1 \leqq |w| \leqq \frac{9}{4}$, $\pi/3 \leqq \theta \leqq 2\pi/3$.

In Cartesian coordinates we have $z = x + iy$ and

$$u = \text{Re}\,(z^2) = x^2 - y^2, \qquad v = \text{Im}\,(z^2) = 2xy.$$

Hence vertical lines $x = c = \text{const}$ are mapped onto $u = c^2 - y^2$, $v = 2cy$. From this we can eliminate y. We obtain $y^2 = c^2 - u$ and $v^2 = 4c^2y^2$. Together,

$$v^2 = 4c^2(c^2 - u) \qquad \text{(Fig. 379).}$$

These parabolas open to the left. Similarly, horizontal lines $y = k = \text{const}$ are mapped onto parabolas opening to the right,

$$v^2 = 4k^2(k^2 + u) \qquad \text{(Fig. 379).}$$ ■

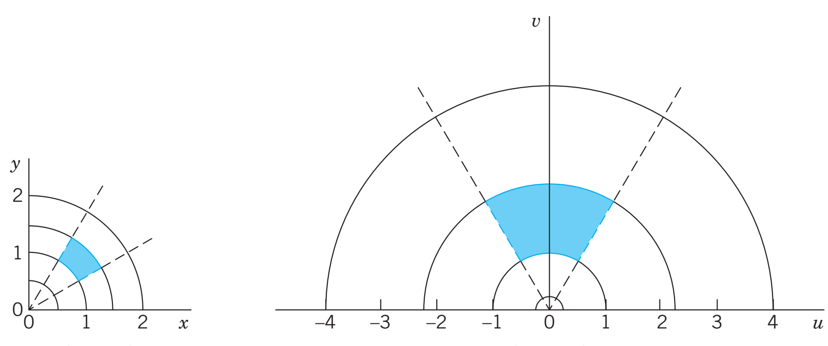


Fig. 378. Mapping $w = z^2$. Lines $|z| = \text{const}$, $\arg z = \text{const}$ and their images in the w-plane

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[1]The general terminology is as follows. A mapping of a set A into a set B is called **surjective** or a mapping of A **onto** B if every element of B is the image of at least one element of A. It is called **injective** or **one-to-one** if different elements of A have different images in B. Finally, it is called **bijective** if it is both surjective and injective.

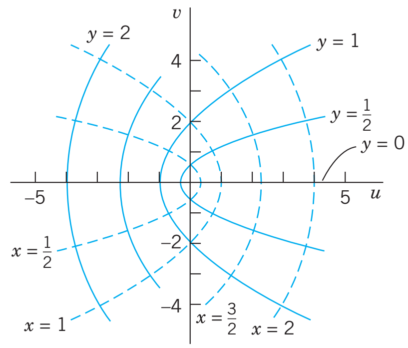


Fig. 379. Images of x = const, y = const under $w = z^2$

Conformal Mapping

A mapping $w = f(z)$ is called **conformal** if it preserves angles between oriented curves in magnitude as well as in sense. Figure 380 shows what this means. The **angle** α $(0 \leqq \alpha \leqq \pi)$ between two intersecting curves C_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0. And *conformality* means that the images C_1^* and C_2^* of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.

THEOREM 1

Conformality of Mapping by Analytic Functions

The mapping $w = f(z)$ by an analytic function f is conformal, except at **critical points**, *that is, points at which the derivative f' is zero.*

PROOF $w = z^2$ has a critical point at $z = 0$, where $f'(z) = 2z = 0$ and the angles are doubled (see Fig. 378), so that conformality fails.

The idea of proof is to consider a curve

$$C: \; z(t) = x(t) + iy(t) \tag{2}$$

in the domain of $f(z)$ and to show that $w = f(z)$ rotates all tangents at a point z_0 (where $f'(z_0) \neq 0$) through the same angle. Now $\dot{z}(t) = dz/dt = \dot{x}(t) + i\dot{y}(t)$ is tangent to C in (2) because this is the limit of $(z_1 - z_0)/\Delta t$ (which has the direction of the secant $z_1 - z_0$

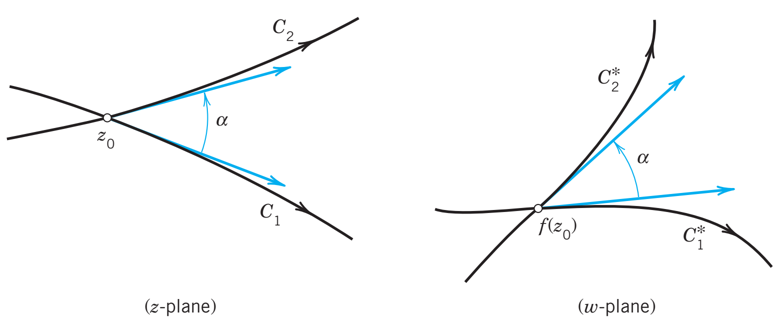


Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

in Fig. 381) as z_1 approaches z_0 along C. The image C^* of C is $w = f(z(t))$. By the chain rule, $\dot{w} = f'(z(t))\dot{z}(t)$. Hence the tangent direction of C^* is given by the argument (use (9) in Sec. 13.2)

$$\arg \dot{w} = \arg f' + \arg \dot{z} \tag{3}$$

where $\arg \dot{z}$ gives the tangent direction of C. This shows that the mapping rotates *all* directions at a point z_0 in the domain of analyticity of f through the same angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$. But this means conformality, as Fig. 381 illustrates for an angle α between two curves, whose images C_1^* and C_2^* make the same angle (because of the rotation). ■

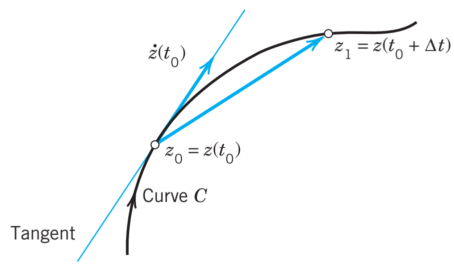


Fig. 381. Secant and tangent of the curve C

In the remainder of this section and in the next ones we shall consider various conformal mappings that are of practical interest, for instance, in modeling potential problems.

EXAMPLE 2 Conformality of $w = z^n$

The mapping $w = z^n$, $n = 2, 3, \cdots$, is conformal, except at $z = 0$, where $w' = nz^{n-1} = 0$. For $n = 2$ this is shown in Fig. 378; we see that at 0 the angles are doubled. For general n the angles at 0 are multiplied by a factor n under the mapping. Hence the sector $0 \leqq \theta \leqq \pi/n$ is mapped by z^n onto the upper half-plane $v \geqq 0$ (Fig. 382). ■

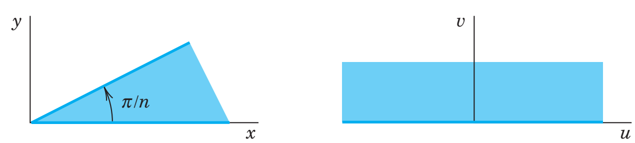


Fig. 382. Mapping by $w = z^n$

EXAMPLE 3 Mapping $w = z + 1/z$. Joukowski Airfoil

In terms of polar coordinates this mapping is

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta).$$

By separating the real and imaginary parts we thus obtain

$$u = a\cos\theta, \qquad v = b\sin\theta \qquad \text{where} \qquad a = r + \frac{1}{r}, \qquad b = r - \frac{1}{r}.$$

Hence circles $|z| = r = \text{const} \neq 1$ are mapped onto ellipses $x^2/a^2 + y^2/b^2 = 1$. The circle $r = 1$ is mapped onto the segment $-2 \leqq u \leqq 2$ of the u-axis. See Fig. 383.

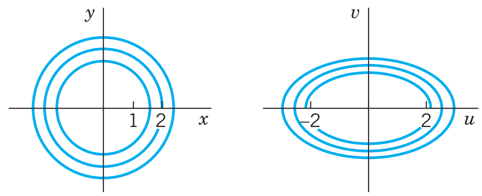


Fig. 383. Example 3

Now the derivative of w is

$$w' = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2}$$

which is 0 at $z = \pm 1$. These are the points at which the mapping is not conformal. The two circles in Fig. 384 pass through $z = -1$. The larger is mapped onto a *Joukowski airfoil.* The dashed circle passes through both -1 and 1 and is mapped onto a curved segment.

Another interesting application of $w = z + 1/z$ (the flow around a cylinder) will be considered in Sec. 18.4. ■

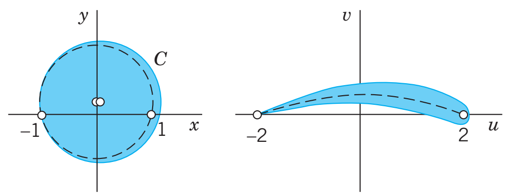


Fig. 384. Joukowski airfoil

EXAMPLE 4 Conformality of $w = e^z$

From (10) in Sec. 13.5 we have $|e^z| = e^x$ and Arg $z = y$. Hence e^z maps a vertical straight line $x = x_0 =$ const onto the circle $|w| = e^{x_0}$ and a horizontal straight line $y = y_0 =$ const onto the ray arg $w = y_0$. The rectangle in Fig. 385 is mapped onto a region bounded by circles and rays as shown.

The fundamental region $-\pi < \text{Arg } z \leqq \pi$ of e^z in the z-plane is mapped bijectively and conformally onto the entire w-plane without the origin $w = 0$ (because $e^z = 0$ for no z). Figure 386 shows that the upper half $0 < y \leqq \pi$ of the fundamental region is mapped onto the upper half-plane $0 < \arg w \leqq \pi$, the left half being mapped inside the unit disk $|w| \leqq 1$ and the right half outside (why?). ■

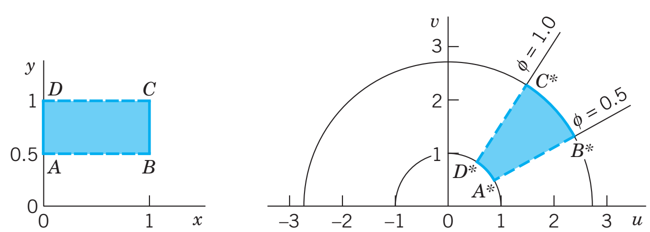


Fig. 385. Mapping by $w = e^z$

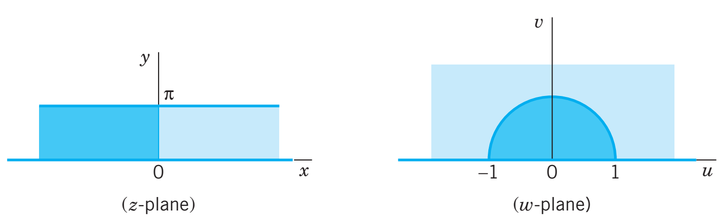


Fig. 386. Mapping by $w = e^z$

EXAMPLE 5 Principle of Inverse Mapping. Mapping $w = \text{Ln}\, z$

Principle. *The mapping by the inverse $z = f^{-1}(w)$ of $w = f(z)$ is obtained by interchanging the roles of the z-plane and the w-plane in the mapping by $w = f(z)$.*

Now the principal value $w = f(z) = \text{Ln}\, z$ of the natural logarithm has the inverse $z = f^{-1}(w) = e^w$. From Example 4 (with the notations z and w interchanged!) we know that $f^{-1}(w) = e^w$ maps the fundamental region of the exponential function onto the z-plane without $z = 0$ (because $e^w \neq 0$ for every w). Hence $w = f(z) = \text{Ln}\, z$ maps the z-plane without the origin and cut along the negative real axis (where $\theta = \text{Im Ln}\, z$ jumps by 2π) conformally onto the horizontal strip $-\pi < v \leqq \pi$ of the w-plane, where $w = u + iv$.

Since the mapping $w = \text{Ln}\, z + 2\pi i$ differs from $w = \text{Ln}\, z$ by the translation $2\pi i$ (vertically upward), this function maps the z-plane (cut as before and 0 omitted) onto the strip $\pi < v \leqq 3\pi$. Similarly for each of the infinitely many mappings $w = \ln z = \text{Ln}\, z \pm 2n\pi i$ ($n = 0, 1, 2, \cdots$). The corresponding horizontal strips of width 2π (images of the z-plane under these mappings) together cover the whole w-plane without overlapping. ■

Magnification Ratio. By the definition of the derivative we have

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|. \tag{4}$$

Therefore, the mapping $w = f(z)$ magnifies (or shortens) the lengths of short lines by approximately the factor $|f'(z_0)|$. The image of a small figure ***conforms*** to the original figure in the sense that it has approximately the same shape. However, since $f'(z)$ varies from point to point, a *large* figure may have an image whose shape is quite different from that of the original figure.

More on the Condition $f'(z) \neq 0$. From (4) in Sec. 13.4 and the Cauchy–Riemann equations we obtain

$$|f'(z)|^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \tag{5'}$$

that is,

$$|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}. \tag{5}$$

This determinant is the so-called **Jacobian** (Sec. 10.3) of the transformation $w = f(z)$ written in real form $u = u(x, y)$, $v = v(x, y)$. Hence $f'(z_0) \neq 0$ implies that the Jacobian is not 0 at z_0. This condition is sufficient that the mapping $w = f(z)$ in a sufficiently small neighborhood of z_0 is one-to-one or injective (different points have different images). See Ref. [GenRef4] in App. 1.

PROBLEM SET 17.1

1. **On Fig. 378.** One "rectangle" and its image are colored. Identify the images for the other "rectangles."
2. **On Example 1.** Verify all calculations.
3. **Mapping $w = z^3$.** Draw an analog of Fig. 378 for $w = z^3$.
4. **Conformality.** Why do the images of the straight lines $x = \text{const}$ and $y = \text{const}$ under a mapping by an analytic function intersect at right angles? Same question for the curves $|z| = \text{const}$ and $\arg z = \text{const}$. Are there exceptional points?

5. Experiment on $w = \bar{z}$. Find out whether $w = \bar{z}$ preserves angles in size as well as in sense. Try to prove your result.

6–9 MAPPING OF CURVES

Find and sketch or graph the images of the given curves under the given mapping.

6. $x = 1, 2, 3, 4, \quad y = 1, 2, 3, 4, \quad w = z^2$
7. **Rotation.** Curves as in Prob. 6, $w = iz$
8. **Reflection in the unit circle.** $|z| = \frac{1}{3}, \frac{1}{2}, 1, 2, 3,$ $\text{Arg } z = 0, \pm\pi/4, \pm\pi/2, \pm 3\pi/2$
9. **Translation.** Curves as in Prob. 6, $w = z + 2 + i$

10. **CAS EXPERIMENT. Orthogonal Nets.** Graph the orthogonal net of the two families of level curves $\text{Re} f(z) = \text{const}$ and $\text{Im} f(z) = \text{const}$, where **(a)** $f(z) = z^4$, **(b)** $f(z) = 1/z$, **(c)** $f(z) = 1/z^2$, **(d)** $f(z) = (z + i)/(1 + iz)$. Why do these curves generally intersect at right angles? In your work, experiment to get the best possible graphs. Also do the same for other functions of your own choice. Observe and record shortcomings of your CAS and means to overcome such deficiencies.

11–20 MAPPING OF REGIONS

Sketch or graph the given region and its image under the given mapping.

11. $|z| \leqq \frac{1}{2}, \quad -\pi/8 < \text{Arg } z < \pi/8, \quad w = z^2$
12. $1 < |z| < 3, \quad 0 < \text{Arg } z < \pi/2, \quad w = z^3$
13. $2 \leqq \text{Im } z \leqq 5, \quad w = iz$
14. $x \geqq 1, \quad w = 1/z$
15. $|z - \frac{1}{2}| \leqq \frac{1}{2}, \quad w = 1/z$
16. $|z| < \frac{1}{2}, \quad \text{Im } z > 0, \quad w = 1/z$
17. $-\text{Ln } 2 \leqq x \leqq \text{Ln } 4, \quad w = e^z$
18. $-1 \leqq x \leqq 2, \quad -\pi < y < \pi, \quad w = e^z$
19. $1 < |z| < 4, \quad \pi/4 < \theta \leqq 3\pi/4, \quad w = \text{Ln } z$
20. $\frac{1}{2} \leqq |z| \leqq 1, \quad 0 \leqq \theta < \pi/2, \quad w = \text{Ln } z$

21–26 FAILURE OF CONFORMALITY

Find all points at which the mapping is not conformal. Give reason.

21. A cubic polynomial
22. $z^2 + 1/z^2$
23. $\dfrac{z + \frac{1}{2}}{4z^2 + 2}$
24. $\exp(z^5 - 80z)$
25. $\cosh z$
26. $\sin \pi z$
27. **Magnification of Angles.** Let $f(z)$ be analytic at z_0. Suppose that $f'(z_0) = 0, \cdots, f^{(k-1)}(z_0) = 0$. Then the mapping $w = f(z)$ magnifies angles with vertex at z_0 by a factor k. Illustrate this with examples for $k = 2, 3, 4$.
28. Prove the statement in Prob. 27 for general $k = 1, 2, \cdots$. *Hint.* Use the Taylor series.

29–35 MAGNIFICATION RATIO, JACOBIAN

Find the magnification ratio M. Describe what it tells you about the mapping. Where is $M = 1$? Find the Jacobian J.

29. $w = \frac{1}{2} z^2$
30. $w = z^3$
31. $w = 1/z$
32. $w = 1/z^2$
33. $w = e^z$
34. $w = \dfrac{z + 1}{2z - 2}$
35. $w = \text{Ln } z$

17.2 Linear Fractional Transformations (Möbius Transformations)

Conformal mappings can help in modeling and solving boundary value problems by first mapping regions conformally onto another. We shall explain this for standard regions (disks, half-planes, strips) in the next section. For this it is useful to know properties of special basic mappings. Accordingly, let us begin with the following very important class.

The next two sections discuss linear fractional transformations. The reason for our thorough study is that such transformations are useful in modeling and solving boundary value problems, as we shall see in Chapter 18. The task is to get a good grasp of which conformal mappings map certain regions conformally onto each other, such as, say mapping a disk onto a half-plane (Sec. 17.3) and so forth. Indeed, the first step in the modeling process of solving boundary value problems is to identify the correct conformal mapping that is related to the "geometry" of the boundary value problem.

The following class of conformal mappings is very important. **Linear fractional transformations** (or **Möbius transformations**) are mappings

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

where a, b, c, d are complex or real numbers. Differentiation gives

$$w' = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2}. \tag{2}$$

This motivates our requirement $ad - bc \neq 0$. It implies conformality for all z and excludes the totally uninteresting case $w' \equiv 0$ once and for all. Special cases of (1) are

$$\begin{aligned} &w = z + b && (\textit{Translations}) \\ &w = az \quad \text{with } |a| = 1 && (\textit{Rotations}) \\ &w = az + b && (\textit{Linear transformations}) \\ &w = 1/z && (\textit{Inversion in the unit circle}). \end{aligned} \tag{3}$$

EXAMPLE 1 Properties of the Inversion $w = 1/z$ (Fig. 387)

In polar forms $z = re^{i\theta}$ and $w = Re^{i\phi}$ the inversion $w = 1/z$ is

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \qquad \text{and gives} \qquad R = \frac{1}{r}, \quad \phi = -\theta.$$

Hence the unit circle $|z| = r = 1$ is mapped onto the unit circle $|w| = R = 1$; $w = e^{i\phi} = e^{-i\theta}$. For a general z the image $w = 1/z$ can be found geometrically by marking $|w| = R = 1/r$ on the segment from 0 to z and then reflecting the mark in the real axis. (Make a sketch.)

Figure 387 shows that $w = 1/z$ maps horizontal and vertical straight lines onto circles or straight lines. Even the following is true.

$w = 1/z$ maps every straight line or circle onto a circle or straight line.

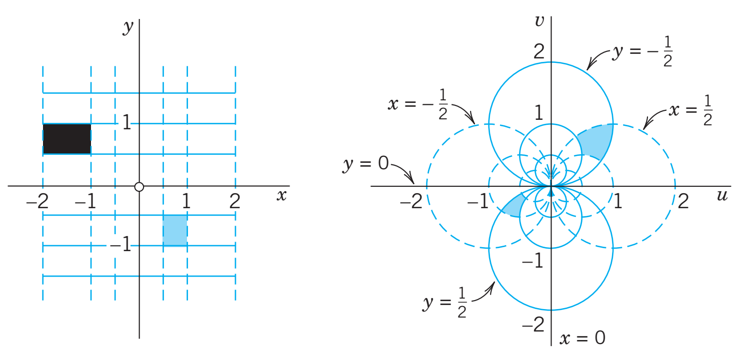


Fig. 387. Mapping (Inversion) $w = 1/z$

Proof. Every straight line or circle in the z-plane can be written

$$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad (A, B\ C, D \text{ real}).$$

$A = 0$ gives a straight line and $A \neq 0$ a circle. In terms of z and $\bar{z}$ this equation becomes

$$Az\bar{z} + B\frac{z + \bar{z}}{2} + C\frac{z - \bar{z}}{2i} + D = 0.$$

Now $w = 1/z$. Substitution of $z = 1/w$ and multiplication by $w\bar{w}$ gives the equation

$$A + B\frac{\bar{w} + w}{2} + C\frac{\bar{w} - w}{2i} + Dw\bar{w} = 0$$

or, in terms of u and v,

$$A + Bu - Cv + D(u^2 + v^2) = 0.$$

This represents a circle (if $D \neq 0$) or a straight line (if $D = 0$) in the w-plane. ■

The proof in this example suggests the use of z and $\bar{z}$ instead of x and y, a ***general principle*** that is often quite useful in practice.

Surprisingly, *every* linear fractional transformation has the property just proved:

THEOREM 1

Circles and Straight Lines

Every linear fractional transformation (1) *maps the totality of circles and straight lines in the z-plane onto the totality of circles and straight lines in the w-plane.*

PROOF This is trivial for a translation or rotation, fairly obvious for a uniform expansion or contraction, and true for $w = 1/z$, as just proved. Hence it also holds for composites of these special mappings. Now comes the key idea of the proof: represent (1) in terms of these special mappings. When $c = 0$, this is easy. When $c \neq 0$, the representation is

$$w = K\frac{1}{cz + d} + \frac{a}{c} \qquad \text{where} \qquad K = -\frac{ad - bc}{c}.$$

This can be verified by substituting K, taking the common denominator and simplifying; this yields (1). We can now set

$$w_1 = cz, \qquad w_2 = w_1 + d, \qquad w_3 = \frac{1}{w_2}, \qquad w_4 = Kw_3,$$

and see from the previous formula that then $w = w_4 + a/c$. This tells us that (1) is indeed a composite of those special mappings and completes the proof. ■

Extended Complex Plane

The extended complex plane (the complex plane together with the point ∞ in Sec. 16.2) can now be motivated even more naturally by linear fractional transformations as follows.

To each z for which $cz + d \neq 0$ there corresponds a unique w in (1). Now let $c \neq 0$. Then for $z = -d/c$ we have $cz + d = 0$, so that no w corresponds to this z. This suggests that we let $w = \infty$ be the image of $z = -d/c$.

Also, the **inverse mapping** of (1) is obtained by solving (1) for z; this gives again a linear fractional transformation

$$z = \frac{dw - b}{-cw + a}. \tag{4}$$

When $c \neq 0$, then $cw - a = 0$ for $w = a/c$, and we let a/c be the image of $z = \infty$. With these settings, the linear fractional transformation (1) is now a one-to-one mapping of the extended z-plane onto the extended w-plane. We also say that every linear fractional transformation maps "the extended complex plane in a one-to-one manner onto itself."

Our discussion suggests the following.

General Remark. If $z = \infty$, then the right side of (1) becomes the meaningless expression $(a \cdot \infty + b)/(c \cdot \infty + d)$. We assign to it the value $w = a/c$ if $c \neq 0$ and $w = \infty$ if $c = 0$.

Fixed Points

Fixed points of a mapping $w = f(z)$ are points that are mapped onto themselves, are "kept fixed" under the mapping. Thus they are obtained from

$$w = f(z) = z.$$

The **identity mapping** $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points, $w = 1/z$ has two, a rotation has one, and a translation none in the finite plane. (Find them in each case.) For (1), the fixed-point condition $w = z$ is

$$z = \frac{az + b}{cz + d}, \qquad \text{thus} \qquad cz^2 - (a - d)z - b = 0. \tag{5}$$

For $c \neq 0$ this is a quadratic equation in z whose coefficients all vanish if and only if the mapping is the identity mapping $w = z$ (in this case, $a = d \neq 0$, $b = c = 0$). Hence we have

THEOREM 2

Fixed Points

A linear fractional transformation, not the identity, has at most two fixed points. If a linear fractional transformation is known to have three or more fixed points, it must be the identity mapping $w = z$.

To make our present general discussion of linear fractional transformations even more useful from a practical point of view, we extend it by further facts and typical examples, in the problem set as well as in the next section.

PROBLEM SET 17.2

1. Verify the calculations in the proof of Theorem 1, including those for the case $c = 0$.
2. **Composition of LFTs.** Show that substituting a linear fractional transformation (LFT) into an LFT gives an LFT.
3. **Matrices.** If you are familiar with 2×2 matrices, prove that the coefficient matrices of (1) and (4) are inverses of each other, provided that $ad - bc = 1$, and that the composition of LFTs corresponds to the multiplication of the coefficient matrices.

4. Fig. 387. Find the image of $x = k = \text{const}$ under $w = 1/z$. *Hint.* Use formulas similar to those in Example 1.

5. Inverse. Derive (4) from (1) and conversely.

6. Fixed points. Find the fixed points mentioned in the text before formula (5).

7–10 INVERSE

Find the inverse $z = z(w)$. Check by solving $z(w)$ for w.

7. $w = \dfrac{i}{2z - 1}$

8. $w = \dfrac{z - i}{z + i}$

9. $w = \dfrac{z - i}{3iz + 4}$

10. $w = \dfrac{z - \frac{1}{2}i}{-\frac{1}{2}iz - 1}$

11–16 FIXED POINTS

Find the fixed points.

11. $w = (a + ib)z^2$

12. $w = z - 3i$

13. $w = 16z^5$

14. $w = az + b$

15. $w = \dfrac{iz + 4}{2z - 5i}$

16. $w = \dfrac{aiz - 1}{z + ai}, \quad a \neq 1$

17–20 FIXED POINTS

Find all LFTs with fixed point(s).

17. $z = 0$

18. $z = \pm 1$

19. $z = \pm i$

20. Without any fixed points

17.3 Special Linear Fractional Transformations

We continue our study of linear fractional transformations. We shall identify linear fractional transformations

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

that map certain standard domains onto others. Theorem 1 (below) will give us a tool for constructing desired linear fractional transformations.

A mapping (1) is determined by a, b, c, d, actually by the ratios of three of these constants to the fourth because we can drop or introduce a common factor. This makes it plausible that three conditions determine a unique mapping (1):

THEOREM 1

Three Points and Their Images Given

Three given distinct points z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one, and only one, linear fractional transformation $w = f(z)$. This mapping is given implicitly by the equation

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}. \tag{2}$$

(If one of these points is the point ∞, the quotient of the two differences containing this point must be replaced by 1.*)*

PROOF Equation (2) is of the form $F(w) = G(z)$ with linear fractional F and G. Hence $w = F^{-1}(G(z)) = f(z)$, where F^{-1} is the inverse of F and is linear fractional (see (4) in

Sec. 17.2) and so is the composite $F^{-1}(G(z))$ (by Prob. 2 in Sec. 17.2), that is, $w = f(z)$ is linear fractional. Now if in (2) we set $w = w_1, w_2, w_3$ on the left and $z = z_1, z_2, z_3$ on the right, we see that

$$F(w_1) = 0, \qquad F(w_2) = 1, \qquad F(w_3) = \infty$$

$$G(z_1) = 0, \qquad G(z_2) = 1, \qquad G(z_3) = \infty.$$

From the first column, $F(w_1) = G(z_1)$, thus $w_1 = F^{-1}(G(z_1)) = f(z_1)$. Similarly, $w_2 = f(z_2)$, $w_3 = f(z_3)$. This proves the existence of the desired linear fractional transformation.

To prove uniqueness, let $w = g(z)$ be a linear fractional transformation, which also maps z_j onto w_j, $j = 1, 2, 3$. Thus $w_j = g(z_j)$. Hence $g^{-1}(w_j) = z_j$, where $w_j = f(z_j)$. Together, $g^{-1}(f(z_j)) = z_j$, a mapping with the three fixed points z_1, z_2, z_3. By Theorem 2 in Sec. 17.2, this is the identity mapping, $g^{-1}(f(z)) = z$ for all z. Thus $f(z) = g(z)$ for all z, the uniqueness.

The last statement of Theorem 1 follows from the General Remark in Sec. 17.2. ■

Mapping of Standard Domains by Theorem 1

Using Theorem 1, we can now find linear fractional transformations of some practically useful domains (here called "standard domains") according to the following principle.

Principle. Prescribe three boundary points z_1, z_2, z_3 of the domain D in the z-plane. Choose their images w_1, w_2, w_3 on the boundary of the image D^* of D in the w-plane. Obtain the mapping from (2). Make sure that D is mapped onto D^*, not onto its complement. In the latter case, interchange two w-points. (Why does this help?)

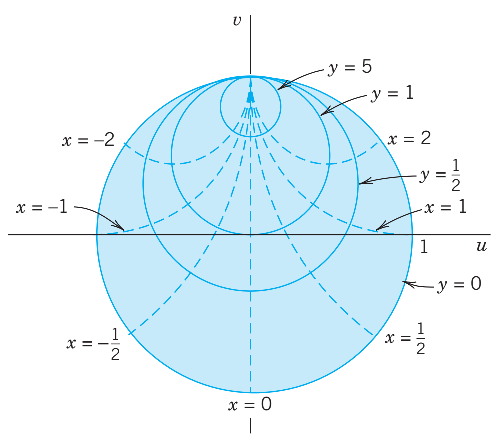


Fig. 388. Linear fractional transformation in Example 1

EXAMPLE 1 Mapping of a Half-Plane onto a Disk (Fig. 388)

Find the linear fractional transformation (1) that maps $z_1 = -1, z_2 = 0, z_3 = 1$ onto $w_1 = -1, w_2 = -i$, $w_3 = 1$, respectively.

Solution. From (2) we obtain

$$\frac{w-(-1)}{w-1} \cdot \frac{-i-1}{-i-(-1)} = \frac{z-(-1)}{z-1} \cdot \frac{0-1}{0-(-1)},$$

thus

$$w = \frac{z - i}{-iz + 1}.$$

Let us show that we can determine the specific properties of such a mapping without much calculation. For $z = x$ we have $w = (x - i)/(-ix + 1)$, thus $|w| = 1$, so that the x-axis maps onto the unit circle. Since $z = i$ gives $w = 0$, the upper half-plane maps onto the interior of that circle and the lower half-plane onto the exterior. $z = 0, i, \infty$ go onto $w = -i, 0, i$, so that the positive imaginary axis maps onto the segment S: $u = 0, -1 \leqq v \leqq 1$. The vertical lines $x = \text{const}$ map onto circles (by Theorem 1, Sec. 17.2) through $w = i$ (the image of $z = \infty$) and perpendicular to $|w| = 1$ (by conformality; see Fig. 388). Similarly, the horizontal lines $y = \text{const}$ map onto circles through $w = i$ and perpendicular to S (by conformality). Figure 388 gives these circles for $y \geqq 0$, and for $y < 0$ they lie outside the unit disk shown. ■

EXAMPLE 2 Occurrence of ∞

Determine the linear fractional transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = -1, w_2 = -i$, $w_3 = 1$, respectively.

Solution. From (2) we obtain the desired mapping

$$w = \frac{z - i}{z + i}.$$

This is sometimes called the *Cayley transformation.*[2] In this case, (2) gave at first the quotient $(1 - \infty)/(z - \infty)$, which we had to replace by 1. ■

EXAMPLE 3 Mapping of a Disk onto a Half-Plane

Find the linear fractional transformation that maps $z_1 = -1, z_2 = i, z_3 = 1$ onto $w_1 = 0, w_2 = i, w_3 = \infty$, respectively, such that the unit disk is mapped onto the right half-plane. (Sketch disk and half-plane.)

Solution. From (2) we obtain, after replacing $(i - \infty)/(w - \infty)$ by 1,

$$w = -\frac{z + 1}{z - 1}.$$ ■

Mapping half-planes onto half-planes is another task of practical interest. For instance, we may wish to map the upper half-plane $y \geqq 0$ onto the upper half-plane $v \geqq 0$. Then the x-axis must be mapped onto the u-axis.

EXAMPLE 4 Mapping of a Half-Plane onto a Half-Plane

Find the linear fractional transformation that maps $z_1 = -2, z_2 = 0, z_3 = 2$ onto $w_1 = \infty, w_2 = \frac{1}{4}, w_3 = \frac{3}{8}$, respectively.

Solution. You may verify that (2) gives the mapping function

$$w = \frac{z + 1}{2z + 4}.$$

What is the image of the x-axis? Of the y-axis? ■

Mappings of disks onto disks is a third class of practical problems. We may readily verify that the unit disk in the z-plane is mapped onto the unit disk in the w-plane by the following function, which maps z_0 onto the center $w = 0$.

---

[2]ARTHUR CAYLEY (1821–1895), English mathematician and professor at Cambridge, is known for his important work in algebra, matrix theory, and differential equations.

(3) $$w = \frac{z - z_0}{cz - 1}, \qquad c = \bar{z}_0, \qquad |z_0| < 1.$$

To see this, take $|z| = 1$, obtaining, with $c = \bar{z}_0$ as in (3),

$$\begin{aligned} |z - z_0| &= |\bar{z} - c| \\ &= |z|\,|\bar{z} - c| \\ &= |z\bar{z} - cz| = |1 - cz| = |cz - 1|. \end{aligned}$$

Hence

$$|w| = |z - z_0|/|cz - 1| = 1$$

from (3), so that $|z| = 1$ maps onto $|w| = 1$, as claimed, with z_0 going onto 0, as the numerator in (3) shows.

Formula (3) is illustrated by the following example. Another interesting case will be given in Prob. 17 of Sec. 18.2.

EXAMPLE 5 Mapping of the Unit Disk onto the Unit Disk

Taking $z_0 = \frac{1}{2}$ in (3), we obtain (verify!)

$$w = \frac{2z - 1}{z - 2} \qquad \text{(Fig. 389).}$$ ■

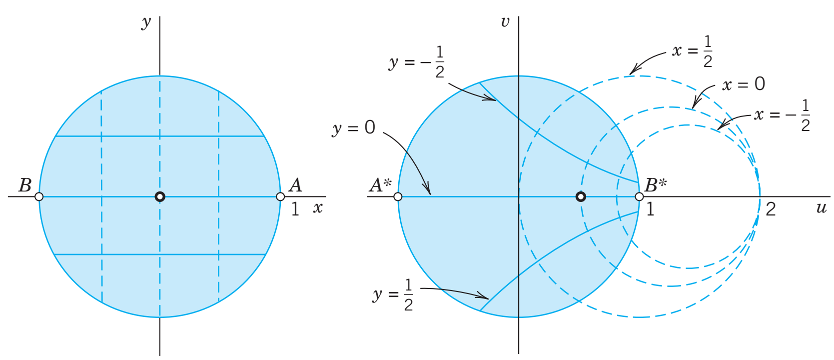


Fig. 389. Mapping in Example 5

EXAMPLE 6 Mapping of an Angular Region onto the Unit Disk

Certain mapping problems can be solved by combining linear fractional transformations with others. For instance, to map the angular region D: $-\pi/6 \leqq \arg z \leqq \pi/6$ (Fig. 390) onto the unit disk $|w| \leqq 1$, we may map D by $Z = z^3$ onto the right Z-half-plane and then the latter onto the disk $|w| \leqq 1$ by

$$w = i\,\frac{Z - 1}{Z + 1}, \qquad \text{combined} \qquad w = i\,\frac{z^3 - 1}{z^3 + 1}.$$ ■

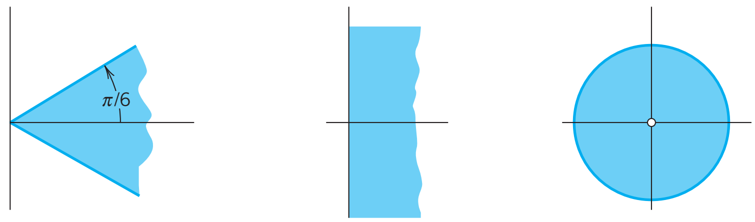


Fig. 390. Mapping in Example 6

This is the end of our discussion of linear fractional transformations. In the next section we turn to conformal mappings by other analytic functions (sine, cosine, etc.).

PROBLEM SET 17.3

1. **CAS EXPERIMENT. Linear Fractional Transformations (LFTs). (a)** Graph typical regions (squares, disks, etc.) and their images under the LFTs in Examples 1–5 of the text.

   (b) Make an experimental study of the continuous dependence of LFTs on their coefficients. For instance, change the LFT in Example 4 continuously and graph the changing image of a fixed region (applying animation if available).
2. **Inverse.** Find the inverse of the mapping in Example 1. Show that under that inverse the lines $x = \text{const}$ are the images of circles in the w-plane with centers on the line $v = 1$.
3. **Inverse.** If $w = f(z)$ is any transformation that has an inverse, prove the (trivial!) fact that f and its inverse have the same fixed points.
4. Obtain the mapping in Example 1 of this section from Prob. 18 in Problem Set 17.2.
5. Derive the mapping in Example 2 from (2).
6. Derive the mapping in Example 4 from (2). Find its inverse and the fixed points.
7. Verify the formula for disks.

8–16 LFTs FROM THREE POINTS AND IMAGES

Find the LFT that maps the given three points onto the three given points in the respective order.

8. $0, 1, 2$ onto $1, \frac{1}{2}, \frac{1}{3}$
9. $1, i, -1$ onto $i, -1, -i$
10. $0, -i, i$ onto $-1, 0, \infty$
11. $-1, 0, 1$ onto $-i, -1, i$
12. $0, 2i, -2i$ onto $-1, 0, \infty$
13. $0, 1, \infty$ onto $\infty, 1, 0$
14. $-1, 0, 1$ onto $1, 1 + i, 1 + 2i$
15. $1, i, 2$ onto $0, -i - 1, -\frac{1}{2}$
16. $-\frac{3}{2}, 0, 1$ onto $0, \frac{3}{2}, 1$
17. Find an LFT that maps $|z| \leqq 1$ onto $|w| \leqq 1$ so that $z = i/2$ is mapped onto $w = 0$. Sketch the images of the lines $x = \text{const}$ and $y = \text{const}$.
18. Find all LFTs $w(z)$ that map the x-axis onto the u-axis.
19. Find an analytic function $w = f(z)$ that maps the region $0 \leqq \arg z \leqq \pi/4$ onto the unit disk $|w| \leqq 1$.
20. Find an analytic function that maps the second quadrant of the z-plane onto the interior of the unit circle in the w-plane.

17.4 Conformal Mapping by Other Functions

We shall now cover mappings by trigonometric and hyperbolic analytic functions. So far we have covered the mappings by z^n and e^z (Sec. 17.1) as well as linear fractional transformations (Secs. 17.2 and 17.3).

Sine Function. Figure 391 shows the mapping by

$$w = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y \qquad (1) \qquad \text{(Sec. 13.6)}.$$

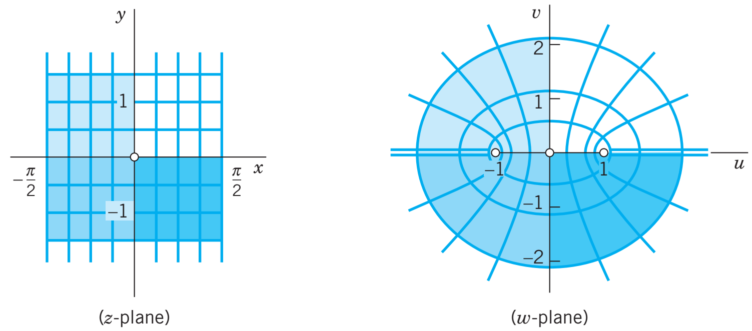


Fig. 391. Mapping $w = u + iv = \sin z$

Hence

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y. \tag{2}$$

Since $\sin z$ is periodic with period 2π, the mapping is certainly not one-to-one if we consider it in the full z-plane. We restrict z to the vertical strip S: $-\frac{1}{2}\pi \leqq x \leqq \frac{1}{2}\pi$ in Fig. 391. Since $f'(z) = \cos z = 0$ at $z = \pm\frac{1}{2}\pi$, the mapping is not conformal at these two critical points. We claim that the rectangular net of straight lines $x = \text{const}$ and $y = \text{const}$ in Fig. 391 is mapped onto a net in the w-plane consisting of hyperbolas (the images of the vertical lines $x = \text{const}$) and ellipses (the images of the horizontal lines $y = \text{const}$) intersecting the hyperbolas at right angles (conformality!). Corresponding calculations are simple. From (2) and the relations $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$ we obtain

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1 \qquad \text{(Hyperbolas)}$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = \sin^2 x + \cos^2 x = 1 \qquad \text{(Ellipses).}$$

Exceptions are the vertical lines $x = -\frac{1}{2}\pi x = \frac{1}{2}\pi$, which are "folded" onto $u \leqq -1$ and $u \geqq 1$ $(v = 0)$, respectively.

Figure 392 illustrates this further. The upper and lower sides of the rectangle are mapped onto semi-ellipses and the vertical sides onto $-\cosh 1 \leqq u \leqq -1$ and $1 \leqq u \leqq \cosh 1$ $(v = 0)$, respectively. An application to a potential problem will be given in Prob. 3 of Sec. 18.2.

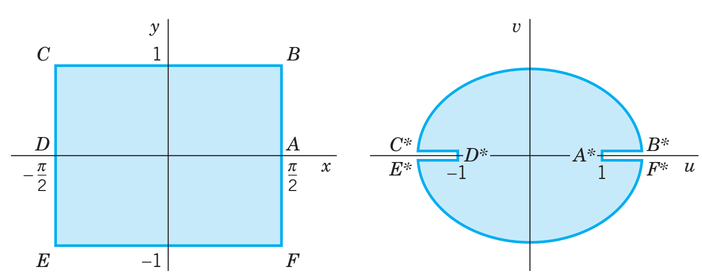


Fig. 392. Mapping by $w = \sin z$

Cosine Function. The mapping $w = \cos z$ could be discussed independently, but since

$$w = \cos z = \sin\left(z + \tfrac{1}{2}\pi\right), \tag{3}$$

we see at once that this is the same mapping as $\sin z$ preceded by a translation to the right through $\frac{1}{2}\pi$ units.

Hyperbolic Sine. Since

$$w = \sinh z = -i \sin (iz), \tag{4}$$

the mapping is a counterclockwise rotation $Z = iz$ through $\frac{1}{2}\pi$ (i.e., 90°), followed by the sine mapping $Z^* = \sin Z$, followed by a clockwise 90°-rotation $w = -iZ^*$.

Hyperbolic Cosine. This function

$$w = \cosh z = \cos (iz) \tag{5}$$

defines a mapping that is a rotation $Z = iz$ followed by the mapping $w = \cos Z$.

Figure 393 shows the mapping of a semi-infinite strip onto a half-plane by $w = \cosh z$. Since $\cosh 0 = 1$, the point $z = 0$ is mapped onto $w = 1$. For real $z = x \geqq 0$, $\cosh z$ is real and increases with increasing x in a monotone fashion, starting from 1. Hence the positive x-axis is mapped onto the portion $u \geqq 1$ of the u-axis.

For pure imaginary $z = iy$ we have $\cosh iy = \cos y$. Hence the left boundary of the strip is mapped onto the segment $1 \geqq u \geqq -1$ of the u-axis, the point $z = \pi i$ corresponding to

$$w = \cosh i\pi = \cos \pi = -1.$$

On the upper boundary of the strip, $y = \pi$, and since $\sin \pi = 0$ and $\cos \pi = -1$, it follows that this part of the boundary is mapped onto the portion $u \leqq -1$ of the u-axis. Hence the boundary of the strip is mapped onto the u-axis. It is not difficult to see that the interior of the strip is mapped onto the upper half of the w-plane, and the mapping is one-to-one.

This mapping in Fig. 393 has applications in potential theory, as we shall see in Prob. 12 of Sec. 18.3.

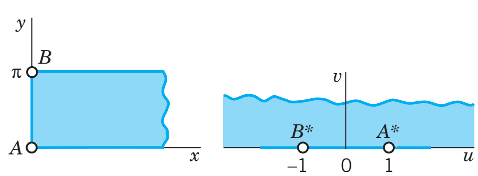


Fig. 393. Mapping by $w = \cosh z$

Tangent Function. Figure 394 shows the mapping of a vertical infinite strip onto the unit circle by $w = \tan z$, accomplished in three steps as suggested by the representation (Sec. 13.6)

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})/i}{e^{iz} + e^{-iz}} = \frac{(e^{2iz} - 1)/i}{e^{2iz} + 1}.$$

Hence if we set $Z = e^{2iz}$ and use $1/i = -i$, we have

$$(6) \qquad w = \tan z = -iW, \qquad W = \frac{Z-1}{Z+1}, \qquad Z = e^{2iz}.$$

We now see that $w = \tan z$ is a linear fractional transformation preceded by an exponential mapping (see Sec. 17.1) and followed by a clockwise rotation through an angle $\frac{1}{2}\pi(90°)$.

The strip is $S\colon -\frac{1}{4}\pi < x < \frac{1}{4}\pi$, and we show that it is mapped onto the unit disk in the w-plane. Since $Z = e^{2iz} = e^{-2y+2ix}$, we see from (10) in Sec. 13.5 that $|Z| = e^{-2y}$, $\operatorname{Arg} Z = 2x$. Hence the vertical lines $x = -\pi/4, 0, \pi/4$ are mapped onto the rays $\operatorname{Arg} Z = -\pi/2, 0, \pi/2$, respectively. Hence S is mapped onto the right Z-half-plane. Also $|Z| = e^{-2y} < 1$ if $y > 0$ and $|Z| > 1$ if $y < 0$. Hence the upper half of S is mapped inside the unit circle $|Z| = 1$ and the lower half of S outside $|Z| = 1$, as shown in Fig. 394.

Now comes the linear fractional transformation in (6), which we denote by $g(Z)$:

$$(7) \qquad W = g(Z) = \frac{Z-1}{Z+1}.$$

For real Z this is real. Hence the real Z-axis is mapped onto the real W-axis. Furthermore, the imaginary Z-axis is mapped onto the unit circle $|W| = 1$ because for pure imaginary $Z = iY$ we get from (7)

$$|W| = |g(iY)| = \left|\frac{iY-1}{iY+1}\right| = 1.$$

The right Z-half-plane is mapped inside this unit circle $|W| = 1$, not outside, because $Z = 1$ has its image $g(1) = 0$ inside that circle. Finally, the unit circle $|Z| = 1$ is mapped onto the imaginary W-axis, because this circle is $Z = e^{i\phi}$, so that (7) gives a pure imaginary expression, namely,

$$g(e^{i\phi}) = \frac{e^{i\phi}-1}{e^{i\phi}+1} = \frac{e^{i\phi/2}-e^{-i\phi/2}}{e^{i\phi/2}+e^{-i\phi/2}} = \frac{i\sin(\phi/2)}{\cos(\phi/2)}.$$

From the W-plane we get to the w-plane simply by a clockwise rotation through $\pi/2$; see (6).

Together we have shown that $w = \tan z$ maps $S\colon -\pi/4 < \operatorname{Re} z < \pi/4$ onto the unit disk $|w| < 1$, with the four quarters of S mapped as indicated in Fig. 394. This mapping is conformal and one-to-one.

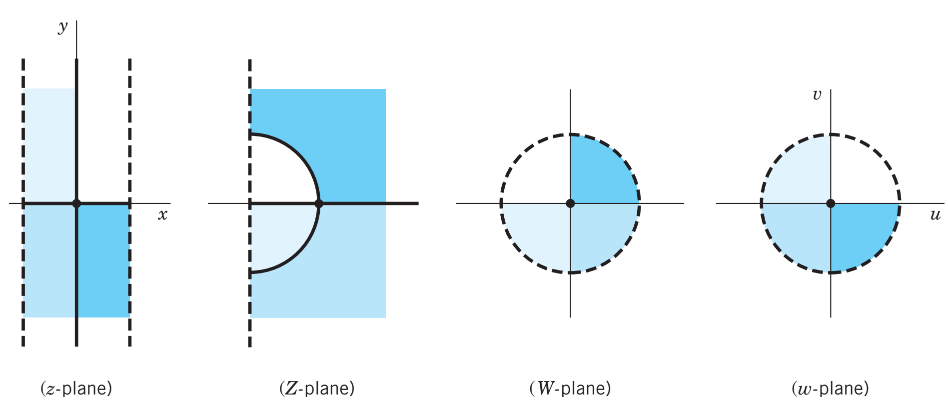


Fig. 394. Mapping by $w = \tan z$

PROBLEM SET 17.4

CONFORMAL MAPPING $w = e^z$

1. Find the image of $x = c = \text{const}$, $-\pi < y \leqq \pi$, under $w = e^z$.
2. Find the image of $y = k = \text{const}$, $-\infty < x \leqq \infty$, under $w = e^z$.

3–7 Find and sketch the image of the given region under $w = e^z$.

3. $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$, $\quad -\pi \leqq y \leqq \pi$
4. $0 < x < 1$, $\quad \frac{1}{2} < y < 1$
5. $-\infty < x < \infty$, $\quad 0 \leqq y \leqq 2\pi$
6. $0 < x < \infty$, $\quad 0 < y < \pi/2$
7. $0 < x < 1$, $\quad 0 < y < \pi$
8. **CAS EXPERIMENT. Conformal Mapping.** If your CAS can do conformal mapping, use it to solve Prob. 7. Then increase y beyond π, say, to 50π or 100π. State what you expected. See what you get as the image. Explain.

CONFORMAL MAPPING $w = \sin z$

9. Find the points at which $w = \sin z$ is not conformal.
10. Sketch or graph the images of the lines $x = 0$, $\pm\pi/6$, $\pm\pi/3$, $\pm\pi/2$ under the mapping $w = \sin z$.

11–14 Find and sketch or graph the image of the given region under $w = \sin z$.

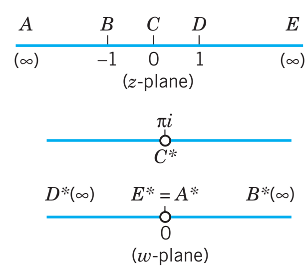
11. $0 < x < \pi/2$, $\quad 0 < y < 2$
12. $-\pi/4 < x < \pi/4$, $\quad 0 < y < 1$
13. $0 < x < 2\pi$, $\quad 1 < y < 3$
14. $0 < x < \pi/6$, $\quad -\infty < y < \infty$
15. Describe the mapping $w = \cosh z$ in terms of the mapping $w = \sin z$ and rotations and translations.
16. Find all points at which the mapping $w = \cosh 2\pi z$ is not conformal.
17. Find an analytic function that maps the region R bounded by the positive x- and y-semi-axes and the hyperbola $xy = \pi$ in the first quadrant onto the upper half-plane. *Hint.* First map R onto a horizontal strip.

CONFORMAL MAPPING $w = \cos z$

18. Find the images of the lines $y = k = \text{const}$ under the mapping $w = \cos z$.
19. Find the images of the lines $x = c = \text{const}$ under the mapping $w = \cos z$.

20–23 Find and sketch or graph the image of the given region under the mapping $w = \cos z$.

20. $0 < x < 2\pi$, $\quad \frac{1}{2} < y < 1$
21. $0 < x < \pi/2$, $\quad 0 < y < 2$ directly and from Prob. 11
22. $-1 < x < 1$, $\quad 0 \leqq y \leqq 1$
23. $\pi < x < 2\pi$, $\quad y < 0$
24. Find and sketch the image of the region $2 \leqq |z| \leqq 3$, $\pi/4 \leqq \theta \leqq \pi/2$ under the mapping $w = \text{Ln}\, z$.
25. Show that $w = \text{Ln}\, \dfrac{z-1}{z+1}$ maps the upper half-plane onto the horizontal strip $0 \leqq \text{Im}\, w \leqq \pi$ as shown in the figure.



Problem 25

17.5 Riemann Surfaces. *Optional*

One of the simplest but most ingenious ideas in complex analysis is that of **Riemann surfaces**. They allow multivalued relations, such as $w = \sqrt{z}$ or $w = \ln z$, to become single-valued and therefore functions in the usual sense. This works because the Riemann surfaces consist of several sheets that are connected at certain points (called branch points). Thus $w = \sqrt{z}$ will need two sheets, being single-valued on each sheet. How many sheets do you think $w = \ln z$ needs? Can you guess, by recalling Sec. 13.7? (The answer will be given at the end of this section). Let us start our systematic discussion.

The mapping given by

$$w = u + iv = z^2 \qquad (1) \quad \text{(Sec. 17.1)}$$

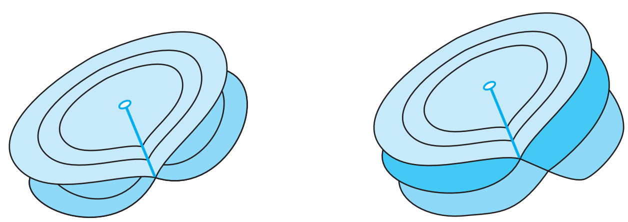
is conformal, except at $z = 0$, where $w' = 2z = 0$, At $z = 0$, angles are doubled under the mapping. Thus the right z-half-plane (including the positive y-axis) is mapped onto the full w-plane, cut along the negative half of the u-axis; this mapping is one-to-one. Similarly for the left z-half-plane (including the negative y-axis). Hence the image of the full z-plane under $w = z^2$ "covers the w-plane twice" in the sense that every $w \neq 0$ is the image of two z-points; if z_1 is one, the other is $-z_1$. For example, $z = i$ and $-i$ are both mapped onto $w = -1$.

Now comes the crucial idea. We place those two copies of the cut w-plane upon each other so that the upper sheet is the image of the right half z-plane R and the lower sheet is the image of the left half z-plane L. We join the two sheets crosswise along the cuts (along the negative u-axis) so that if z moves from R to L, its image can move from the upper to the lower sheet. The two origins are fastened together because $w = 0$ is the image of just one z-point, $z = 0$. The surface obtained is called a **Riemann surface** (Fig. 395a). $w = 0$ is called a "winding point" or **branch point**. $w = z^2$ maps the full z-plane onto this surface in a one-to-one manner.

By interchanging the roles of the variables z and w it follows that the double-valued relation

$$w = \sqrt{z} \qquad (2) \qquad \text{(Sec. 13.2)}$$

becomes single-valued on the Riemann surface in Fig. 395a, that is, a function in the usual sense. We can let the upper sheet correspond to the principal value of $\sqrt{z}$. Its image is the right w-half-plane. The other sheet is then mapped onto the left w-half-plane.



(a) Riemann surface of $\sqrt{z}$ (b) Riemann surface of $\sqrt[3]{z}$

Fig. 395. Riemann surfaces

Similarly, the triple-valued relation $w = \sqrt[3]{z}$ becomes single-valued on the three-sheeted Riemann surface in Fig. 395b, which also has a branch point at $z = 0$.

The infinitely many-valued natural logarithm (Sec. 13.7)

$$w = \ln z = \text{Ln}\, z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

becomes single-valued on a Riemann surface consisting of infinitely many sheets, $w = \text{Ln}\, z$ corresponds to one of them. This sheet is cut along the negative x-axis and the upper edge of the slit is joined to the lower edge of the next sheet, which corresponds to the argument $\pi < \theta \leqq 3\pi$, that is, to

$$w = \text{Ln}\, z + 2\pi i.$$

The principal value Ln z maps its sheet onto the horizontal strip $-\pi < v \leqq \pi$. The function $w = \text{Ln}\, z + 2\pi i$ maps its sheet onto the neighboring strip $\pi < v \leqq 3\pi$, and so on. The mapping of the points $z \neq 0$ of the Riemann surface onto the points of the w-plane is one-to-one. See also Example 5 in Sec. 17.1.

PROBLEM SET 17.5

1. If z moves from $z = \frac{1}{4}$ twice around the circle $|z| = \frac{1}{4}$, what does $w = \sqrt{z}$ do?

2. Show that the Riemann surface of $w = \sqrt{(z-1)(z-2)}$ has branch points at 1 and 2 sheets, which we may cut and join crosswise from 1 to 2. *Hint.* Introduce polar coordinates $z - 1 = r_1 e^{i\theta_1}$ and $z - 2 = r_2 e^{i\theta_2}$, so that $w = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$.

3. Make a sketch, similar to Fig. 395, of the Riemann surface of $w = \sqrt[4]{z} + 1$.

4–10 RIEMANN SURFACES

Find the branch points and the number of sheets of the Riemann surface.

4. $\sqrt{iz - 2 + i}$

5. $z^2 + \sqrt[3]{4z + i}$

6. $\ln (6z - 2i)$

7. $\sqrt[n]{z - z_0}$

8. $e^{\sqrt{z}}, \sqrt{e^z}$

9. $\sqrt{z^3 + z}$

10. $\sqrt{(4 - z^2)(1 - z^2)}$

CHAPTER 17 REVIEW QUESTIONS AND PROBLEMS

1. What is a conformal mapping? Why does it occur in complex analysis?

2. At what points are $w = z^5 - z$ and $w = \cos(\pi z^2)$ not conformal?

3. What happens to angles at z_0 under a mapping $w = f(z)$ if $f'(z_0) = 0, f''(z_0) = 0, f'''(z_0) \neq 0$?

4. What is a linear fractional transformation? What can you do with it? List special cases.

5. What is the extended complex plane? Ways of introducing it?

6. What is a fixed point of a mapping? Its role in this chapter? Give examples.

7. How would you find the image of $x = \text{Re } z = 1$ under $w = iz, z^2, e^z, 1/z$?

8. Can you remember mapping properties of $w = \ln z$?

9. What mapping gave the Joukowski airfoil? Explain details.

10. What is a Riemann surface? Its motivation? Its simplest example.

11–16 MAPPING $w = z^2$

Find and sketch the image of the given region or curve under $w = z^2$.

11. $1 < |z| < 2, \quad |\arg z| < \pi/8$

12. $1/\sqrt{\pi} < |z| < \sqrt{\pi}, \quad 0 < \arg z < \pi/2$

13. $-4 < xy < 4$

14. $0 < y < 2$

15. $x = -1, 1$

16. $y = -2, 2$

17–22 MAPPING $w = 1/z$

Find and sketch the image of the given region or curve under $w = 1/z$.

17. $|z| < 1$

18. $|z| < 1, \quad 0 < \arg z < \pi/2$

19. $2 < |z| < 3, \quad y > 0$

20. $0 \leqq \arg z \leqq \pi/4$

21. $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}, \quad y > 0$

22. $z = 1 + iy \quad (-\infty < y < \infty)$

23–28 LINEAR FRACTIONAL TRANSFORMATIONS (LFTs)

Find the LFT that maps

23. $-1, 0, 1$ onto $4 + 3i, 5i/2, 4 - 3i$, respectively

24. $0, 2, 4$ onto $\infty, \frac{1}{2}, \frac{1}{4}$, respectively

25. $1, i, -i$ onto $i, -1, 1$, respectively

26. $0, 1, 2$ onto $2i, 1 + 2i, 2 + 2i$, respectively

27. $0, 1, \infty$ onto $\infty, 1, 0$, respectively

28. $-1, -i, i$ onto $1 - i, 2, 0$, respectively

29–34 FIXED POINTS

Find the fixed points of the mapping

29. $w = (2 + i)z$

30. $w = z^4 + z - 64$

31. $w = (3z + 2)/(z - 1)$

32. $(2iz - 1)/(z + 2i)$

33. $w = z^5 + 10z^3 + 10z$

34. $w = (iz + 5)/(5z + i)$

35–40 GIVEN REGIONS

Find an analytic function $w = f(z)$ that maps

35. The infinite strip $0 < y < \pi/4$ onto the upper half-plane $v > 0$.

36. The quarter-disk $|z| < 1, x > 0, y > 0$ onto the exterior of the unit circle $|w| = 1$.

37. The sector $0 < \arg z < \pi/2$ onto the region $u < 1$.

38. The interior of the unit circle $|z| = 1$ onto the exterior of the circle $|w + 2| = 2$.

39. The region $x > 0, y > 0, xy < c$ onto the strip $0 < v < 1$.

40. The semi-disk $|z| < 2, y > 0$ onto the exterior of the circle $|w - \pi| = \pi$.

SUMMARY OF CHAPTER 17
Conformal Mapping

A complex function $w = f(z)$ gives a **mapping** of its domain of definition in the complex z-plane onto its range of values in the complex w-plane. If $f(z)$ is *analytic*, this mapping is **conformal**, that is, angle-preserving: the images of any two intersecting curves make the same angle of intersection, in both magnitude and sense, as the curves themselves (Sec. 17.1). Exceptions are the points at which $f'(z) = 0$ ("**critical points**," e.g. $z = 0$ for $w = z^2$).

For mapping properties of e^z, $\cos z$, $\sin z$ etc. see Secs. 17.1 and 17.4.

Linear fractional transformations, also called *Möbius transformations*

$$w = \frac{az + b}{cz + d} \qquad \text{(1)} \qquad \text{(Secs. 17.2, 17.3)}$$

$(ad - bc \neq 0)$ map the extended complex plane (Sec. 17.2) onto itself. They solve the problems of mapping half-planes onto half-planes or disks, and disks onto disks or half-planes. Prescribing the images of three points determines (1) uniquely.

Riemann surfaces (Sec. 17.5) consist of several sheets connected at certain points called *branch points.* On them, multivalued relations become single-valued, that is, functions in the usual sense. *Examples.* For $w = \sqrt{z}$ we need two sheets (with branch point 0) since this relation is doubly-valued. For $w = \ln z$ we need infinitely many sheets since this relation is infinitely many-valued (see Sec. 13.7).