گروه درس: .... نام استاد: .... تاریخ: .... نمره: ....

سوال 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 1\,dx = \frac{1}{\pi}(\pi) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,dx = \frac{1}{n\pi}\sin(nx)\Big|_{-\pi}^{0}$$

$$= \frac{1}{n\pi}(0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx = \frac{1}{\pi}\left(\frac{-1}{n}\cos nx\Big|_{-\pi}^{0}\right.$$

$$= \frac{-1}{n\pi}\left(1-\cos n\pi\right) = \frac{-1}{n\pi}\left(1-(-1)^n\right)$$

$$b_n = \frac{-2}{(2n-1)\pi}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{(2n-1)\pi}\sin(2n-1)x$$

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نام و نام خانوادگی: .... شماره دانشجویی: .... نام درس: ....

گروه درس: .... نام استاد: .... تاریخ: .... نمره: ....

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} b_n^2 = \frac{1}{\pi}\int_{-\pi}^{0} 1\, dx = 1$$

$$\Rightarrow \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} = 1 \Rightarrow$$

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$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{2} \times \frac{1}{4} \times \pi^2 = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

.... نام درس: .... شماره دانشجویی: ....

گروه درس: .... نام استاد: .... تاریخ: .... نمره: ....

سوال ۲

$$\begin{cases} u_t = 9u_{xx} & 0<x<+\infty \quad (1)\\ u(0,t)=0 & (2)\\ x\to+\infty \text{ کراندار و حقیقی } u(x,t) & (3)\\ u(x,0)=f(x), \quad f(x)=\begin{cases}1 & 0<x<1\\ 0 & x\geq 1\end{cases} & (4)\end{cases}$$

$u(x,t)=X(x)T(t)$

$\xrightarrow{(2)} X(0)T(t)=0 \longrightarrow X(0)=0$

$\xrightarrow{(3)}$ $x\to+\infty$ کراندار و حقیقی $X(x)T(t) \longrightarrow$ $x\to+\infty$ کراندار و حقیقی $X(x)$

$\xrightarrow{(1)} \dfrac{T'}{9T}=\dfrac{X''}{X}=-\sigma$ ثابت $\to X''+\sigma X=0, \quad T'=-9\sigma T$

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$\to \begin{cases} X''+\sigma X=0\\ X(0)=0\\ x\to+\infty \text{ کراندار و حقیقی } X(x)\end{cases}$

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احتمال اول $\sigma=-\lambda^2<0 \to X(x)=c_1e^{\lambda x}+c_2e^{-\lambda x} \xrightarrow{X(x) \text{ کراندار}} c_1=0 \to X(x)=c_2e^{-\lambda x}$

$\xrightarrow{X(0)=0} c_2=0 \longrightarrow X(x)=0$ جواب بدیهی

حالت دوم $\sigma=\lambda^2>0 \to X(x)=c_1\cos\lambda x+c_2\sin\lambda x$

$\xrightarrow{X(0)=0} c_1=0 \to X(x)=c_2\sin\lambda x$ کراندار است $\to X_\lambda(x)=\sin\lambda x$

$\forall\lambda>0$

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$T'=-9\sigma T \xrightarrow{\sigma=\lambda^2} T_\lambda(t)=e^{-9\lambda^2t}$

$\therefore u_\lambda(x,t)=e^{-9\lambda^2t}\sin\lambda x, \quad \forall\lambda>0$

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$\to u(x,t)=\int_0^{+\infty}A_\lambda e^{-9\lambda^2t}\sin\lambda x\,d\lambda$

$\xrightarrow{(4)} f(x)=\int_0^{+\infty}A_\lambda\sin\lambda x\,d\lambda \longrightarrow A_\lambda=\frac{2}{\pi}\int_0^{+\infty}f(x)\sin\lambda x\,dx$

$=\frac{2}{\pi}\int_0^1\sin\lambda x\,dx$

$=\frac{2}{\pi}\left[-\frac{\cos\lambda x}{\lambda}\right]_0^1=\frac{2(1-\cos\lambda)}{\pi\lambda}$

$\therefore u(x,t)=\frac{2}{\pi}\int_0^{+\infty}\frac{1-\cos\lambda}{\lambda}e^{-9\lambda^2t}\sin\lambda x\,d\lambda$

۳)

$$\begin{cases} u_{xx} = u_t - x + e^{-t}\sin(\pi x) & 0<x<1 \\ u(0,t)=0 & t>0 \\ u(1,t)=t \\ u(x,0)=\sin(2\pi x) \end{cases}$$

حل:

$$u(x,t) = w(x,t) + v(x,t)$$

$$v(x,t) = a(t)x + b(t) \Longrightarrow v(0,t) = b(t) = 0 \Longrightarrow v(x,t) = xt$$

$$v(1,t) = a(t) = t$$

$$\begin{cases} w_{xx} = w_t + x - x + e^{-t}\sin\pi x \\ w(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = \sin(2\pi x) \end{cases}$$

۲ پایه : $\{\sin(\pi n x)\}_{n=1}^{\infty}$

$$\Longrightarrow w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) \Longrightarrow$$ ۳

$$\sum_{1}^{\infty} -(n\pi)^2 w_n(t)\sin(n\pi x) = \sum_{n=1}^{\infty} \dot{w}_n(t)\sin(n\pi x) + e^{-t}\sin(\pi x)$$

$$\Longrightarrow \begin{cases} \dot{w}_n(t) + (n\pi)^2 w_n(t) = 0 & \forall n \neq 1 \\ \dot{w}_n(t) + (n\pi)^2 w_n(t) = -e^{-t} & n=1 \Longrightarrow \dot{w}_1(t) + \pi^2 w_1(t) = - \end{cases}$$ ۵

$$w_n(t) = e^{-(n\pi)^2 t} w_n(0)$$ ۳

$$w_1(t) = e^{-\pi^2 t} w_1(0) + \int_0^t -e^{-\pi^2(t-s)} e^{-s}\, ds = e^{-\pi^2 t} w_1(0) - e^{-\pi^2 t}$$

$$= e^{-\pi^2 t} w_1(0) - e^{-\pi^2 t} \times \frac{1}{\pi^2 - 1} e^{(\pi^2-1)s}\Big|_0^t = e^{-\pi^2 t} w_1(0) - \frac{e^{-\pi^2 t}}{\pi^2-1}\left[e^{+(\pi^2-1)t} - \right.$$ ۹

$$= e^{-\pi^2 t} w_1(0) - \frac{e^{-t}}{\pi^2 - 1} + \frac{e^{-\pi^2 t}}{\pi^2-1}$$

$$\sin(2\pi x) = w(x,0) = \sum_{n=1}^{\infty} w_n(0)\sin(n\pi x)$$

+4

$$\Rightarrow \begin{cases} w_2(0) = 1 \\ w_n(0) = 0 \qquad \forall\, n \neq 2 \end{cases}$$

$$\Rightarrow w_2(t) = e^{-(4\pi^2 t)} \quad , \quad w_1(t) = \frac{e^{-\pi^2 t}}{\pi^2 - 1} - \frac{e^{-t}}{\pi^2 - 1}$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) = e^{-4\pi^2 t}\sin(2\pi x) + \frac{e^{-\pi^2 t} - e^{-t}}{\pi^2 - 1}\sin\pi x$$

$$\Rightarrow u(x,t) = w + v = e^{-4\pi^2 t}\sin(2\pi x) + \frac{e^{-\pi^2 t} - e^{-t}}{\pi^2 - 1}\sin\pi x + x t$$

بسمه تعالی

بارم سوال ۴ ریاضی مهندسی

جواب به ۱۰ قسمت تقسیم شده و به هر قسمت ۳ نمره اختصاص داده شده است.

$$u(r,\theta)=R(r)\Phi(\theta),$$

$$\begin{cases} u(r,0)=0 \Longrightarrow R(r)\Phi(0)=0 \Longrightarrow \Phi(0)=0. \\ u(1,\theta)=0 \Longrightarrow R(1)\Phi(\theta)=0 \Longrightarrow R(1)=0. \\ u(e,\theta)=0 \Longrightarrow R(e)\Phi(\theta)=0 \Longrightarrow R(e)=0. \end{cases}$$ ۳

$$r^2u_{rr}+ru_r+u_{\theta\theta}=0 \Longrightarrow r^2R''(r)\Phi(\theta)+rR'(r)\Phi(\theta)+R(r)\Phi''(\theta)=0$$

$$\Longrightarrow \frac{r^2R''(r)+rR'(r)}{R(r)}+\frac{\Phi''(\theta)}{\Phi(\theta)}=0$$

$$\Longrightarrow \frac{r^2R''(r)+rR'(r)}{R(r)}=-\frac{\Phi''(\theta)}{\Phi(\theta)}=\lambda.$$ ۳

$$\Longrightarrow r^2R''(r)+rR'(r)-\lambda R(r)=\Phi''(\theta)+\lambda\Phi(\theta)=0.$$

$$\Longrightarrow \begin{cases} r^2R''(r)+rR'(r)-\lambda R(r)=0, \\ R(1)=R(e)=0 \end{cases} \qquad \begin{cases} \Phi''(\theta)+\lambda\Phi(\theta)=0, \\ \Phi(0)=0 \end{cases}$$ ۳

$$R(r)=r^m \Longrightarrow m^2-\lambda=0 \Longrightarrow m_{1,2}=\pm\sqrt{\lambda}.$$

$$(1)\ \lambda=0 \Longrightarrow m_1=m_2=0 \Longrightarrow R(r)=c_1+c_2\ln r.$$
$$\Longrightarrow R(1)=c_1+c_2\ln 1=c_1,\quad R(e)=c_1+c_2\ln e=c_1+c_2.$$
$$R(1)=R(e)=0 \Longrightarrow c_1=c_1+c_2=0 \Longrightarrow c_1=c_2=0$$
$$\Longrightarrow R(r)=0.$$ ۳

$$(2)\ \lambda=k^2>0;\ k>0 \Longrightarrow m_{1,2}=\pm\sqrt{k^2}=\pm k.$$
$$\Longrightarrow R(r)=c_1r^k+c_2r^{-k}=a\cosh(k\ln r)+b\sinh(k\ln r).$$
$$R(1)=0 \Longrightarrow c_1+c_2=a=0 \Longrightarrow a=0, c_2=-c_1.$$
$$R(e)=0 \Longrightarrow c_1e^k+c_2e^{-k}=a\cosh(k)+b\sinh(k)=0.$$
$$\Longrightarrow c_1e^k-c_1e^{-k}=b\sinh(k)=0.$$
$$\Longrightarrow 2c_1\sinh(k)=b\sinh(k)=0 \Longrightarrow 2c_1=b=0 \Longrightarrow c_2=0.$$
$$\Longrightarrow c_1=c_2=a=b=0$$
$$\Longrightarrow R(r)=0.$$ ۳

$$(۳)\ \lambda = -k^{۲} < ۰;\ k > ۰ \Longrightarrow m_{۱,۲} = \pm\sqrt{-k^{۲}} = \pm ik.$$
$$\Longrightarrow R(r) = a\cos(k\ln r) + b\sin(k\ln r).$$
$$R(۱) = ۰ \Longrightarrow a = ۰.$$
$$R(e) = ۰ \Longrightarrow b\sin(k) = ۰ \Longrightarrow b = ۰ \text{ یا } \sin(k) = ۰$$
$$\Longrightarrow R(r) = ۰ \text{ یا } k = n\pi, n = ۱, ۲, ۳, \ldots$$
$$k = n\pi \Longrightarrow R_n(r) = b_n\sin(n\pi\ln r),\quad \lambda = -k^{۲} = -n^{۲}\pi^{۲}.$$

۳

$$\Longrightarrow \begin{cases} \Phi''(\theta) - n^{۲}\pi^{۲}\Phi(\theta) = ۰, \\ \Phi(۰) = ۰ \end{cases} \Longrightarrow \Phi_n(\theta) = \sinh(n\pi\theta).$$

۳

$$\Longrightarrow u_n(r,\theta) = R_n(r)\Phi_n(\theta) = b_n\sin(n\pi\ln r)\sinh(n\pi\theta).$$
$$\Longrightarrow u(r,\theta) = \sum_{n=۱}^{\infty} u_n(r,\theta) = \sum_{n=۱}^{\infty} b_n\sinh(n\pi\theta)\sin(n\pi\ln r).$$

۳

$$u(r,\frac{\pi}{۲}) = \sin(۴\pi\ln r) \Longrightarrow \sin(۴\pi\ln r) = \sum_{n=۱}^{\infty} b_n\sinh(n\pi^{۲}/۲)\sin(n\pi\ln r).$$
$$\Longrightarrow b_۴\sinh(۲\pi^{۲}) = ۱,\quad b_۱ = b_۲ = b_۳ = b_۵ = b_۶ = \cdots = ۰.$$

۳

$$\Longrightarrow u(r,\theta) = b_۴\sinh(۴\pi\theta)\sin(۴\pi\ln r)$$
$$= \frac{۱}{\sinh(۲\pi^{۲})}\sinh(۴\pi\theta)\sin(۴\pi\ln r).$$

۳