Math Behind Blockchain Protocol

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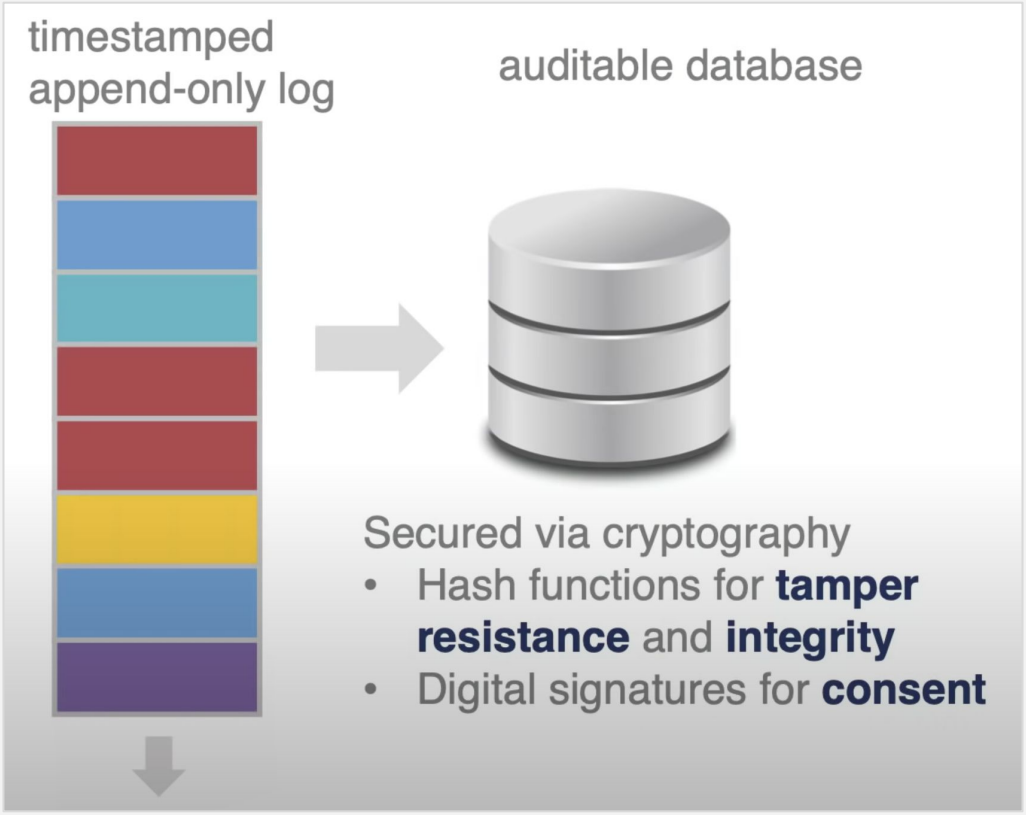
What is Blockchain?



Internet - Open Protocol

- Ethernet 1974
- TCP/IP 1974
- HTTP 1990

1994 

1984 

1979 

Commercialization

‘The Net’ 1994

First Online Sale 1994

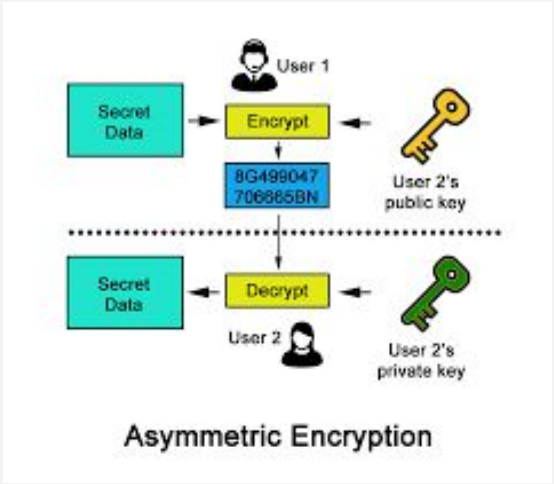
No Online Payment!

Cryptography

Communication in the presence of adversaries

Ancient Cipher

Enigma WWII



Asymmetric Cryptography 1976

SSL/TLS 1996

Digital Currencies Failors

- DigiCash 1989
- Mondex 1993
- CyberCash 1994
- E-gold 1996
- Hashcash 1997
- Bit Gold 1998
- B-Money 1998
- Lurce 1999

Digital Payment Innovation

2003

2007

The Riddle Remained

How to move value

peer-to-peer

without any

Trusted central intermediary

Bitcoin: A Peer-to-Peer Electronic Cash System

The announcement

From: Satoshi Nakamoto <satoshi <at> vistomail.com>
Subject: Bitcoin P2P e-cash paper
Newsgroups: gmane.comp.encryption.general
Date: 2008-10-31 18:10:00 GMT

I've been working on a new electronic cash system that's fully peer-to-peer, with no trusted third party.
The paper is available at: http://www.bitcoin.org/bitcoin.pdf
The main properties:
Double-spending is prevented with a peer-to-peer network.
No mint or other trusted parties.
Participants can be anonymous.
New coins are made from Hashcash style proof-of-work.
The proof-of-work for new coin generation also powers the network to prevent double-spending.

Bitcoin: A Peer-to-Peer Electronic Cash System
Abstract. A purely peer-to-peer version of electronic cash [...]

Satoshi Nakamoto

The Cryptography Mailing List

12/27

2009

Origin

Bitcoin: A Peer-to-Peer Electronic Cash System

Satoshi Nakamoto
satoshin@gmx.com
www.bitcoin.org

Abstract. A purely peer-to-peer version of electronic cash would allow online payments to be sent directly from one party to another without going through a financial institution. Digital signatures provide part of the solution, but the main benefits are lost if a trusted third party is still required to prevent double-spending. We propose a solution to the double-spending problem using a peer-to-peer network. The network timestamps transactions by hashing them into an ongoing chain of hash-based proof-of-work, forming a record that cannot be changed without redoing the proof-of-work. The longest chain not only serves as proof of the sequence of events witnessed, but proof that it came from the largest pool of CPU power. As long as a majority of CPU power is controlled by nodes that are not cooperating to attack the network, they'll generate the longest chain and outpace attackers. The network itself requires minimal structure. Messages are broadcast on a best effort basis, and nodes can leave and rejoin the network at will, accepting the longest proof-of-work chain as proof of what happened while they were gone.

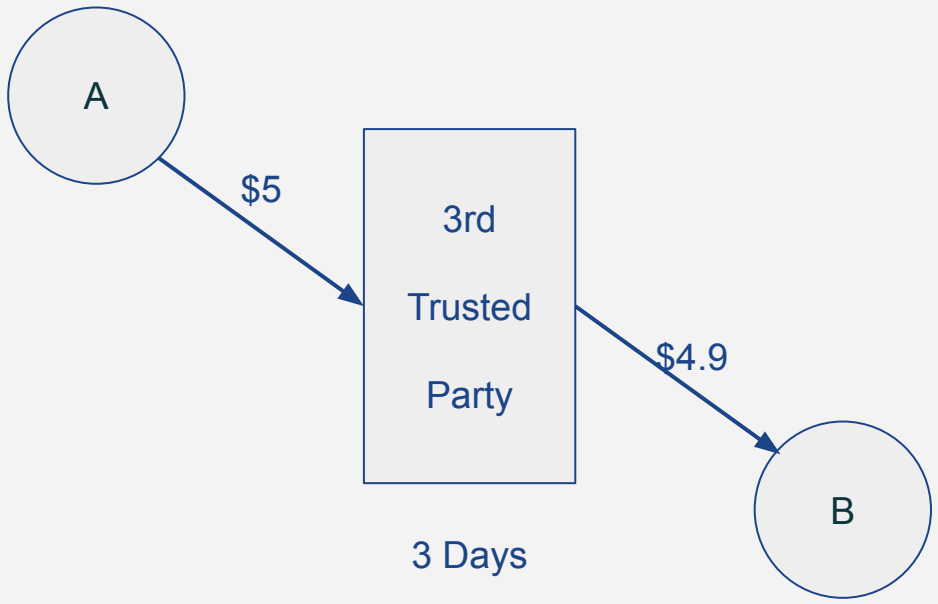
https://bitcoin.org/bitcoin.pdf

Blockchain Technology

- Verifiably moves ‘data’ on a decentralized network
- The ‘data’ can represent value or computer code
- Thus it goes directly to the plumbing financial sector and money
- Broad adoption rests on addressing technical, commercial and public policy hurdle
- It can be a catalyst for change in the world of finance and money

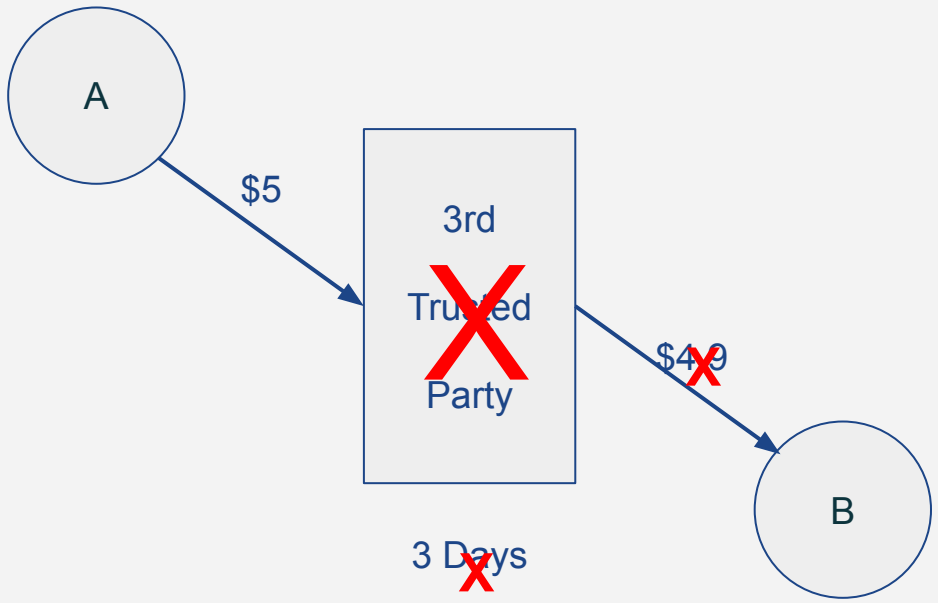
Blockchain: Technology of Money Transfer

- How we transfer money?



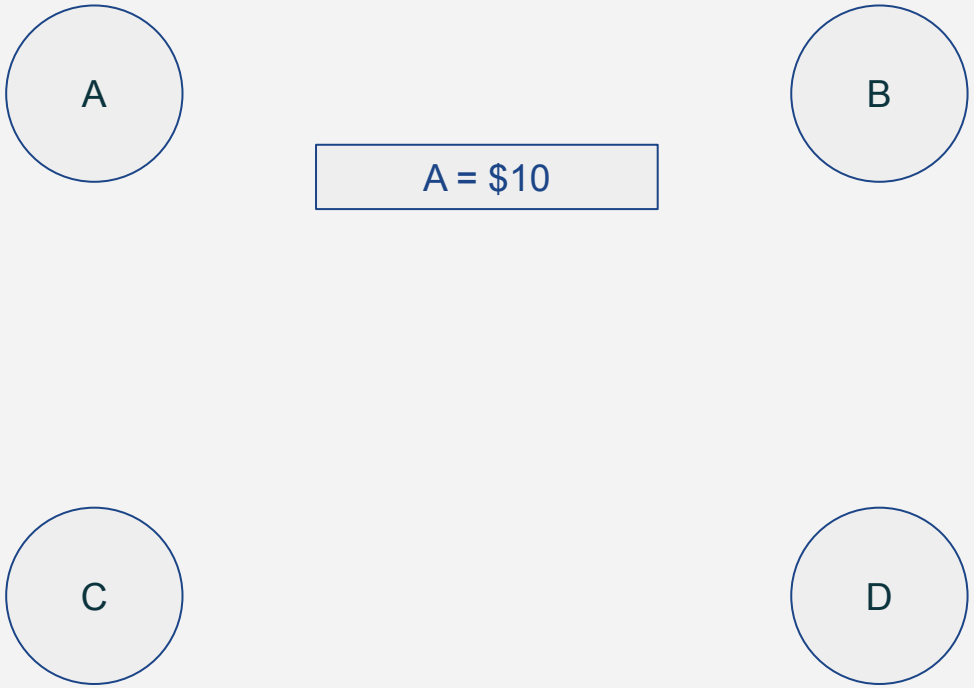
Blockchain: Technology of Money Transfer

- In Blockchain



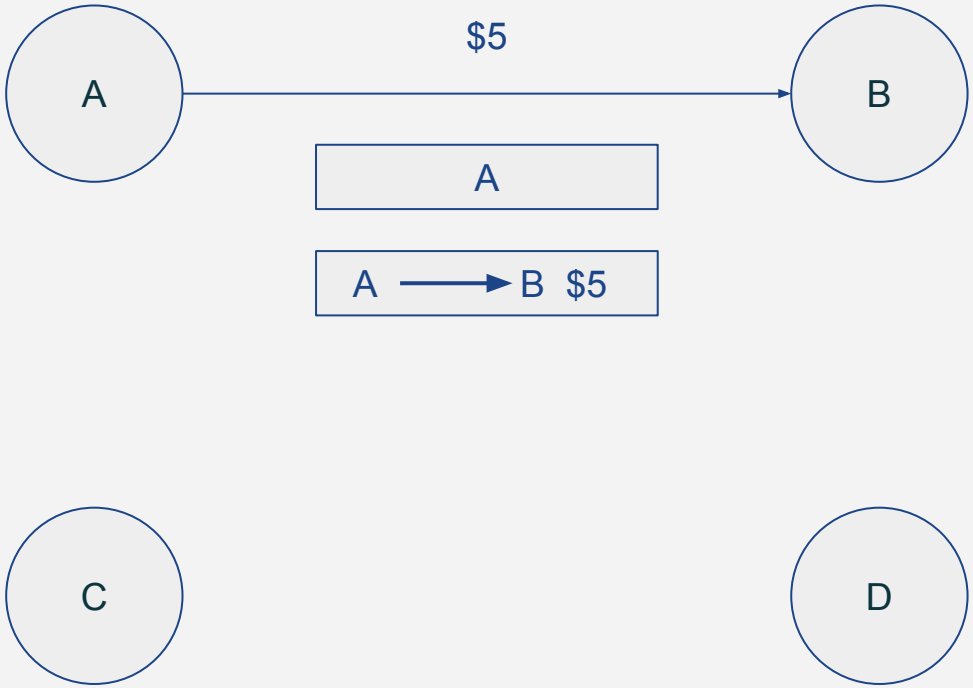
Blockchain: Technology of Money Transfer

- Open Ledger



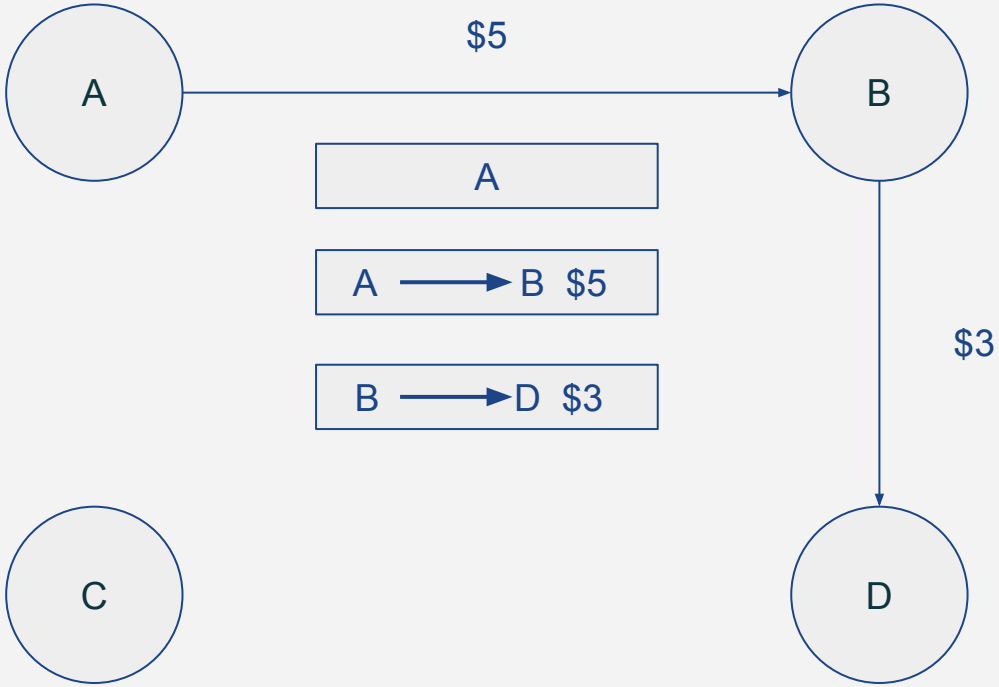
Blockchain: Technology of Money Transfer

- Open Ledger



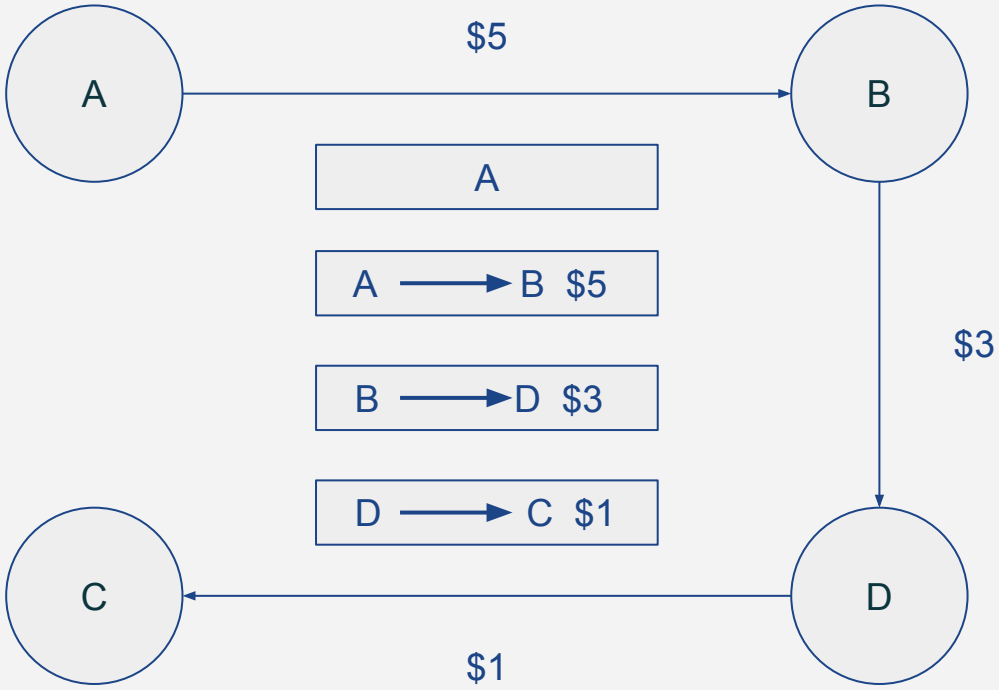
Blockchain: Technology of Money Transfer

- Open Ledger



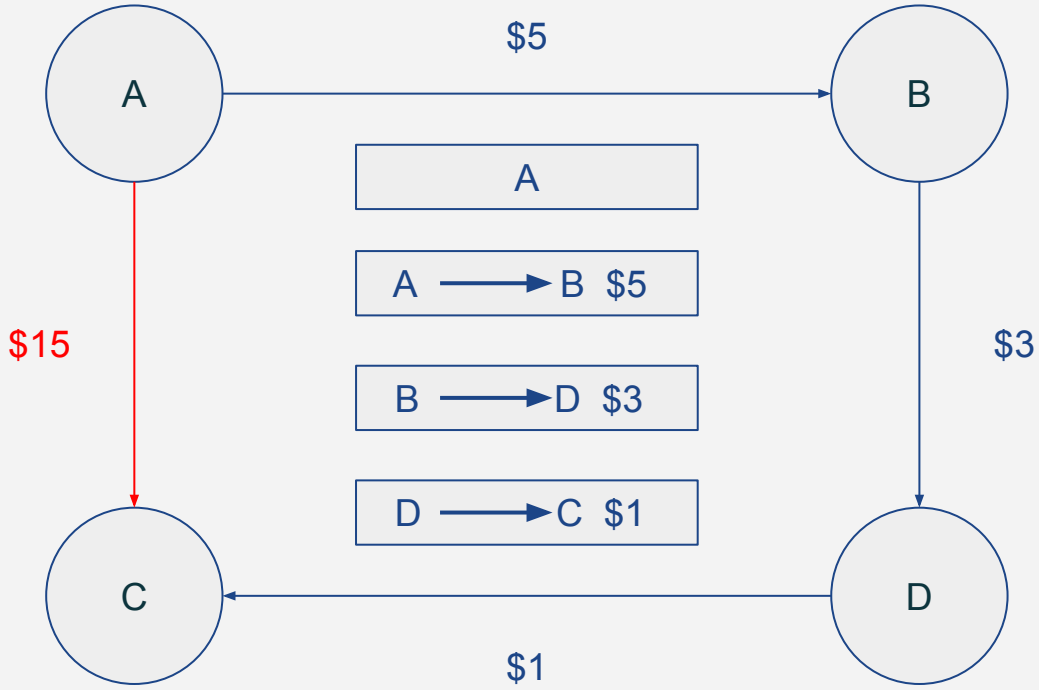
Blockchain: Technology of Money Transfer

- Open Ledger



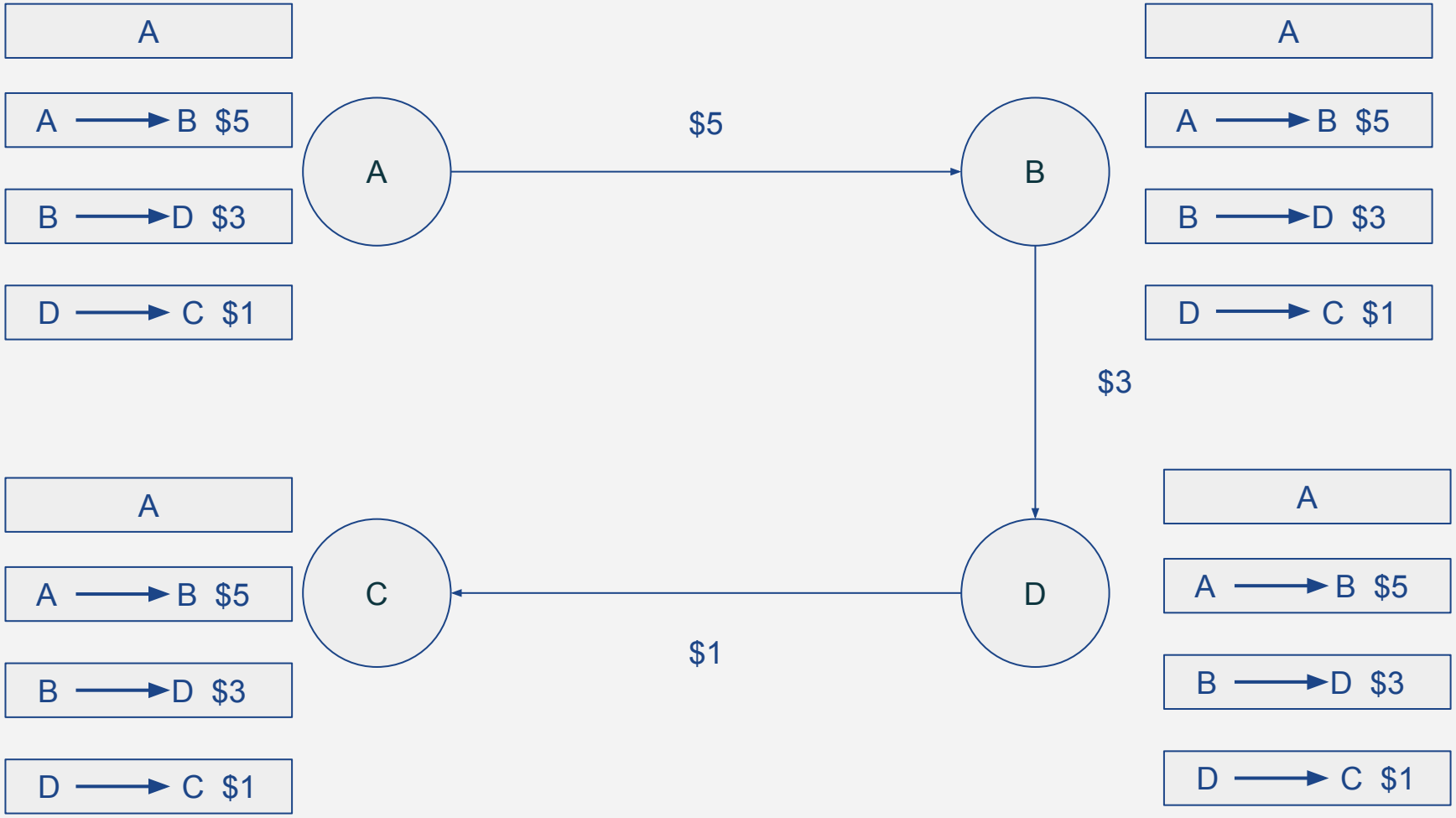
Blockchain: Technology of Money Transfer

- Open Ledger



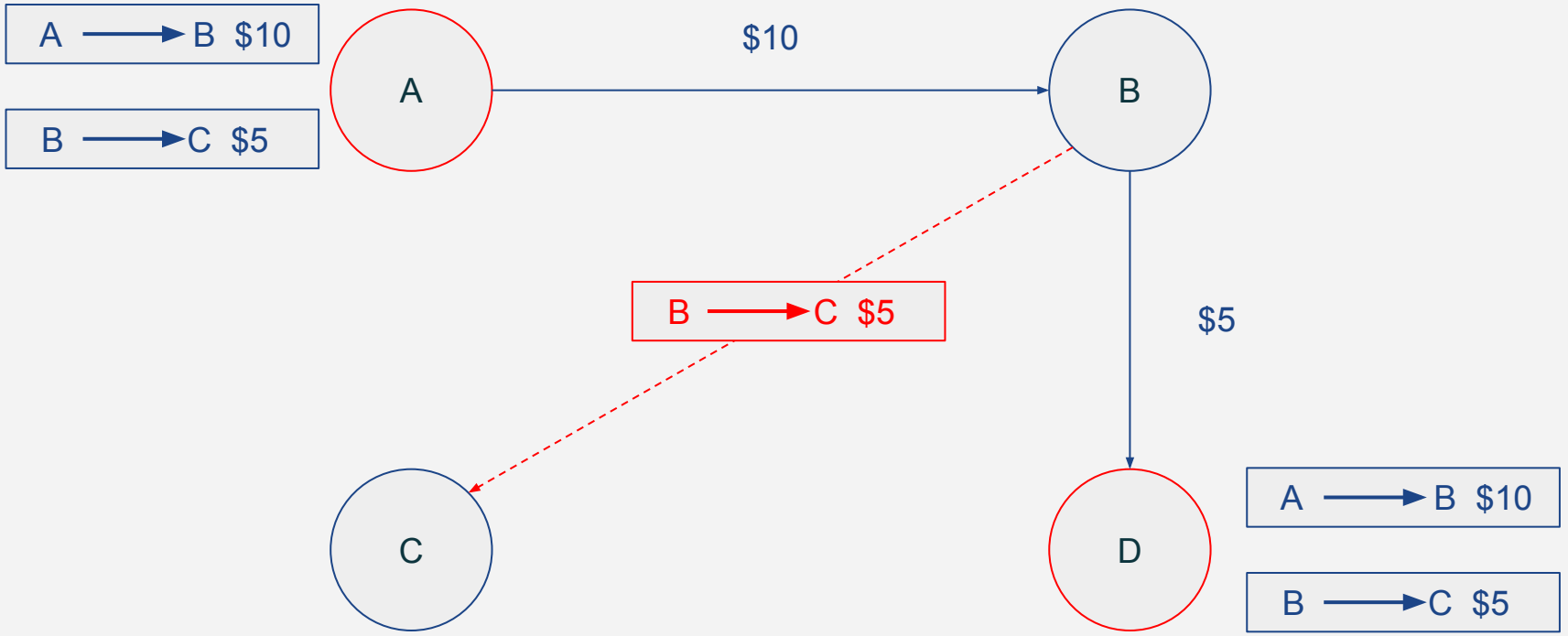
Blockchain: Technology of Money Transfer

- Distributed Open Ledger



Blockchain: Technology of Money Transfer

- Miners

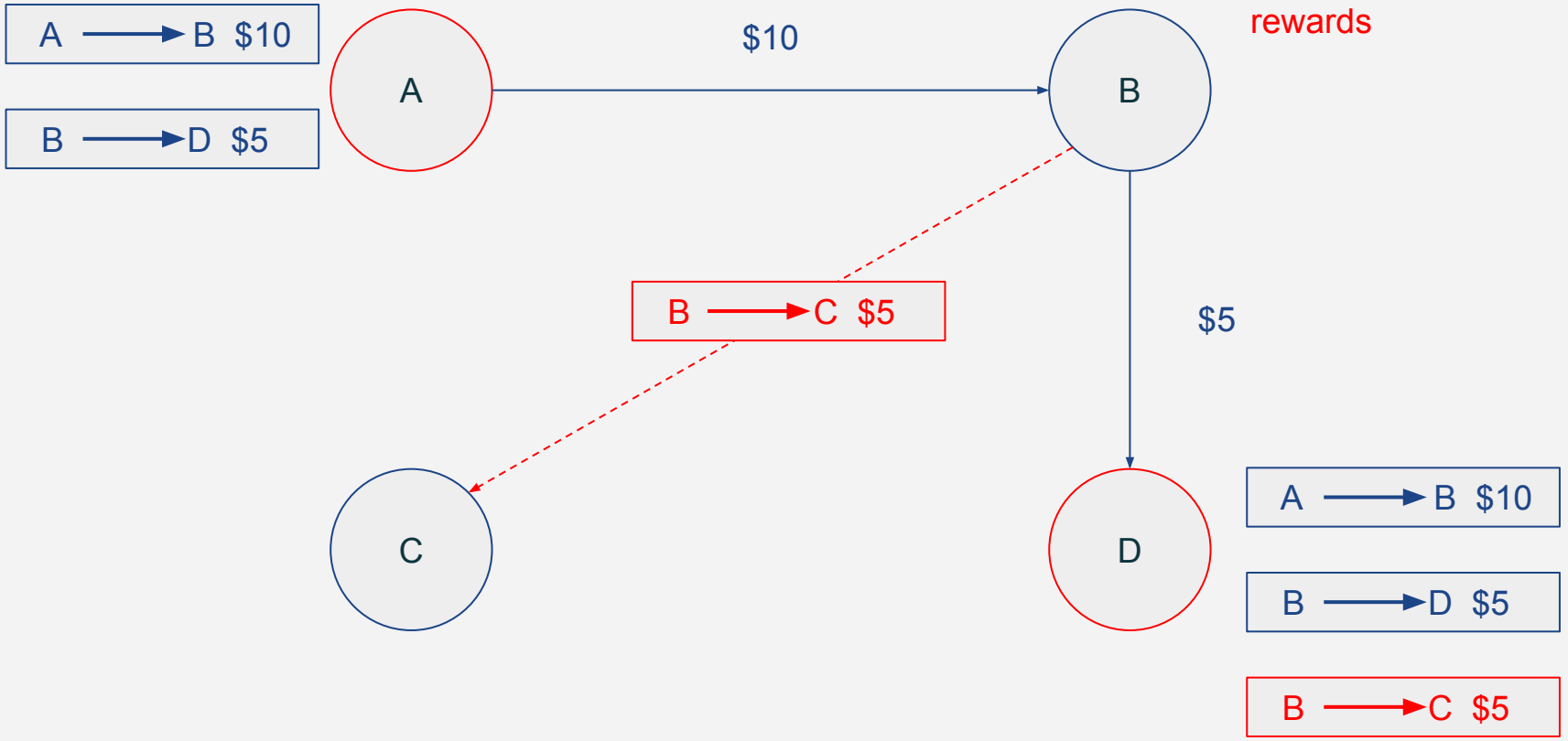


Blockchain: Technology of Money Transfer

- Synchronizing the Ledger - Miners

What miners do?

- Validate
- Key
- First one receive rewards

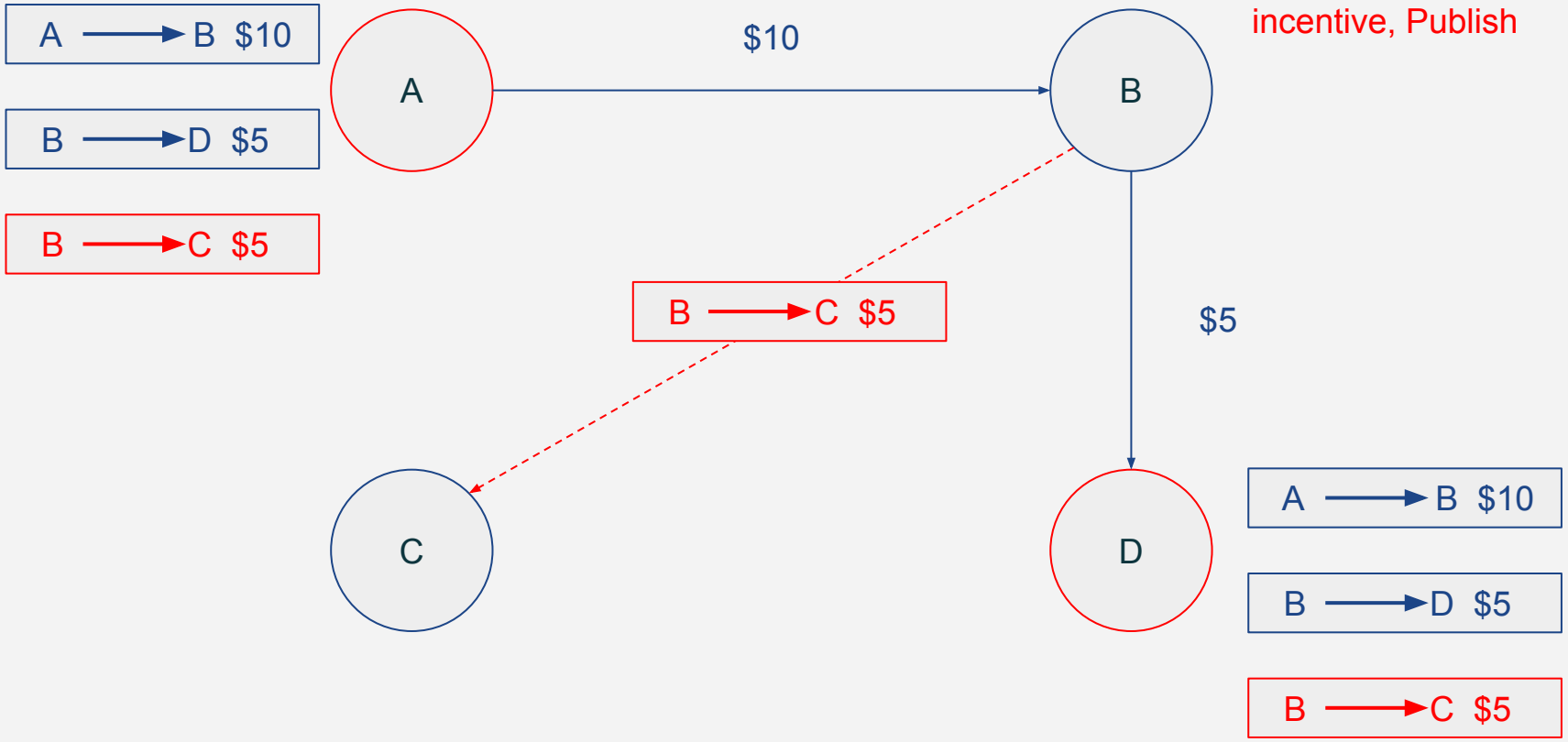


Blockchain: Technology of Money Transfer

- Synchronizing the Ledger - Miners

What miners do?

- Validate
- Key
- First one receive an incentive, Publish

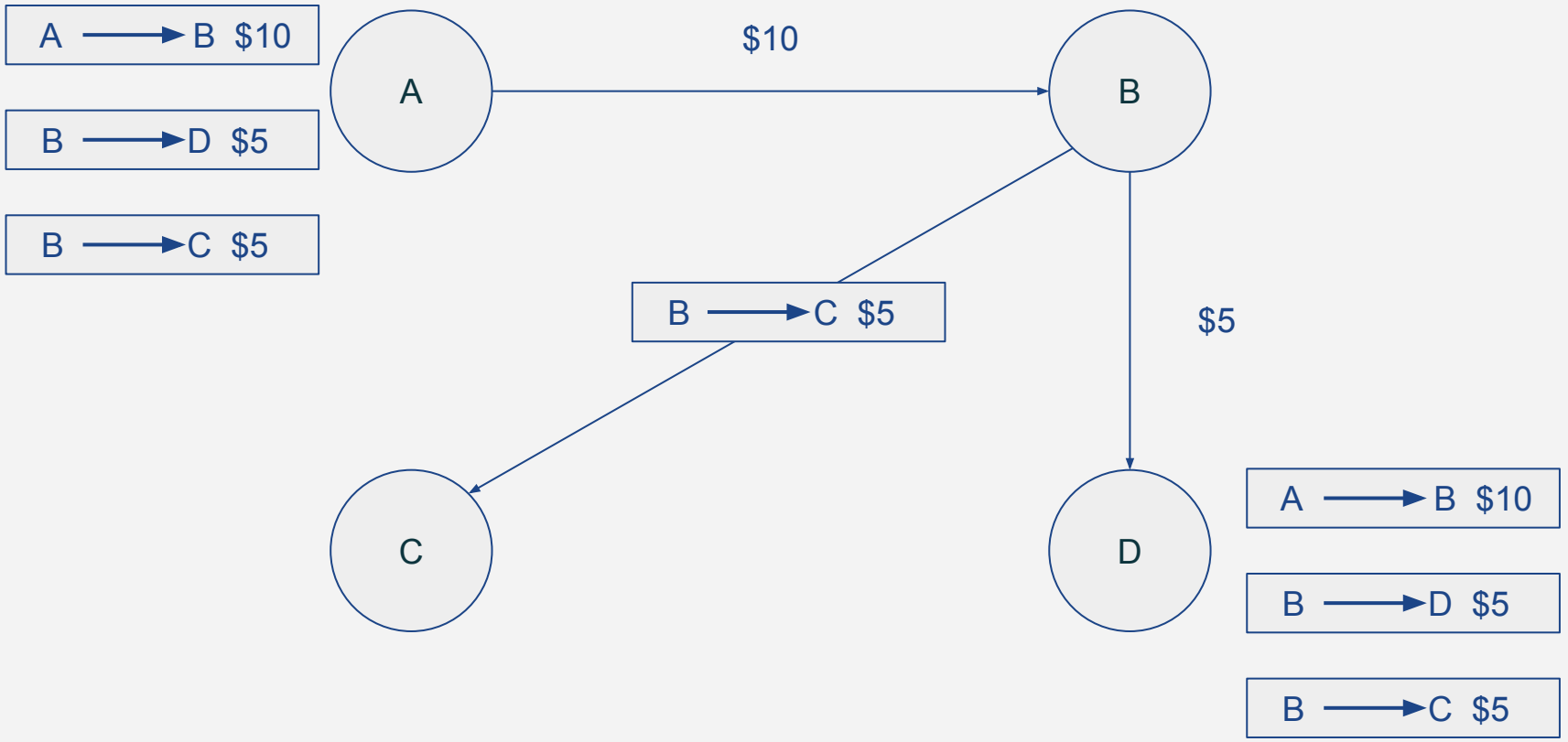


Blockchain: Technology of Money Transfer

- Summary

Principles

- Open/Public
- Distributed
- Miners



Blockchain

- Blockchain is a specific type of database.
- It differs from a typical database in the way it stores information; blockchains store data in blocks that are then chained together.
- As new data comes in it is entered into a fresh block. Once the block is filled with data it is chained onto the previous block, which makes the data chained together in chronological order.
- Different types of information can be stored on a blockchain but the most common use so far has been as a ledger for transactions.
- In Bitcoin's case, blockchain is used in a decentralized way so that no single person or group has control—rather, all users collectively retain control.
- Decentralized blockchains are immutable, which means that the data entered is irreversible. For Bitcoin, this means that transactions are permanently recorded and viewable to anyone.

Bitcoin Protocol

- Bitcoins exist as records on a distributed ledger of blockchain, copies of which are shared by a volunteer network of connected computers.
- To “own” a bitcoin simply means having the ability to transfer control of it to someone else by creating a record of the transfer in the blockchain.
- What grants this ability? Access to an ECDSA private and public key pair.
- What does that mean and how does that secure bitcoin?

  ECDSA is short for **Elliptic Curve Digital Signature Algorithm**. It’s a process that uses an elliptic curve and a finite field to “sign” data in such a way that third parties can verify the authenticity of the signature while the signer retains the exclusive ability to create the signature.

- With bitcoin, the data that is signed is the transaction that transfers ownership.

Procedures

- ECDSA has separate procedures for **signing** and **verification.**
- Each procedure is an algorithm composed of a few arithmetic operations.
- The signing algorithm makes use of the private key, and the verification process makes use of the public key.

We will show an example of this later. But first let’s know about:

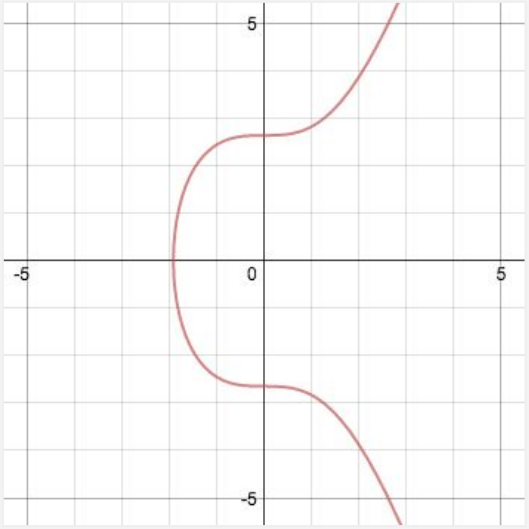
- Elliptic curves
- Finite fields.

Elliptic Curves

An elliptic curve is represented algebraically as an equation of the form:

$y^2 = x^3 + ax + b$

For *a* = *0* and *b* = *7* (the version used by bitcoin), it looks like this:



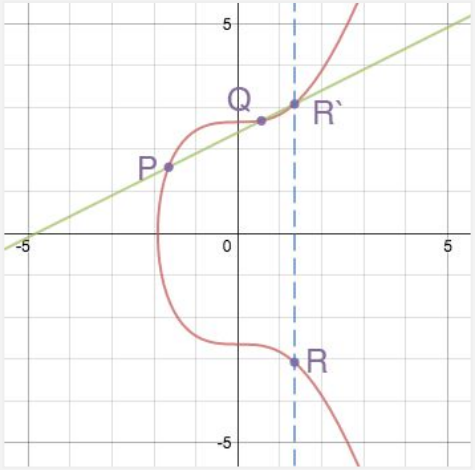
- A non-vertical line intersecting two non-tangent points on the curve will always intersect a third point on the curve.
- A non-vertical line tangent to the curve at one point will intersect precisely one other point on the curve.

We can use these properties to define two operations: **point addition** and **point doubling**.

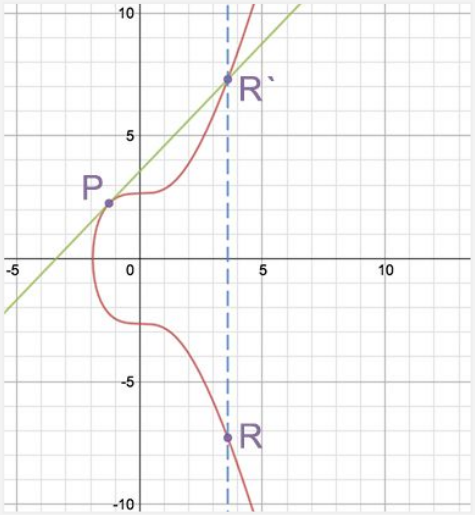
Addition, Doubling

- Point addition, $P + Q = R$, is defined as the reflection through the x-axis of the third intersecting point R' on a line that includes P and Q. Like figure a.

(a)



(b)



- Similarly, point doubling, $P + P = R$ is defined by finding the line tangent to the point to be doubled, P, and taking reflection through the x-axis of the intersecting point R' on the curve to get R.

  Here's an example of what that would look like figure b.

Example

Together, these two operations are used for **scalar multiplication**, $R = a\,P$, defined by adding the point P to itself a times.

R = 7P

R = P + (P + (P + (P + (P + (P + P)))))

The process of scalar multiplication is normally simplified by using a combination of point addition and point doubling operations. For example:

R = 7P

R = P + 6P

R = P + 2 (3P)

R = P + 2 (P + 2P)

Here, *7P* has been broken down into two point doubling steps and two point addition steps.

Finite Field

- A finite field, in the context of ECDSA, can be thought of as a predefined range of positive numbers within which every calculation must fall. Any number outside this range “wraps around” so as to fall within the range.
- The simplest way to think about this is calculating remainders, as represented by the modulus (mod) operator.

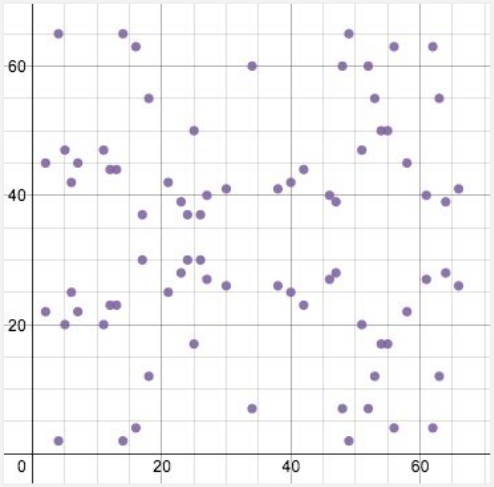
  For example, 9/7 gives 1 with a remainder of 2:

$$9 \bmod 7 = 2$$

- Here our finite field is modulo 7, and all mod operations over this field yield a result falling within a range from 0 to 6.

Example

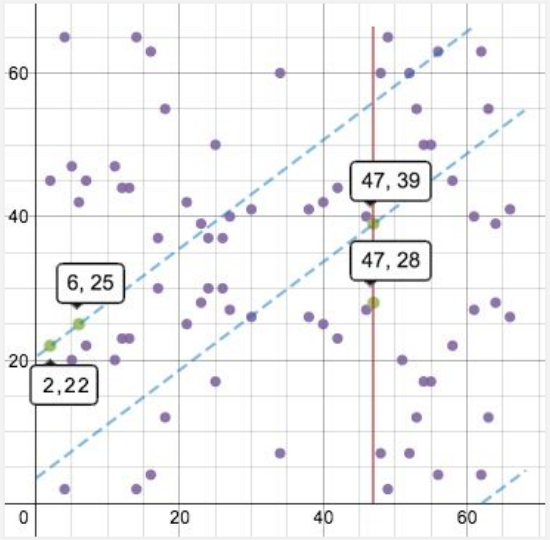
ECDSA uses elliptic curves in the context of a finite field, which greatly changes their appearance but not their underlying equations or special properties. The same equation plotted above, in a finite field of modulo 67, looks like this:



It's now a set of points, in which all the *x* and *y* values are integers between 0 and 66. Note that the "curve" still retains its horizontal symmetry.

Example

Point addition and doubling are now slightly different visually. Lines drawn on this graph will wrap around the horizontal and vertical directions. So adding points (2, 22) and (6, 25) looks like this:



The third intersecting point is (47, 39) and its reflection point is (47, 28).

ECDSA and Bitcoin

- A protocol such as bitcoin selects a set of parameters for the elliptic curve and its finite field representation that is fixed for all users of the protocol.
- The parameters include the **equation** used, the **prime modulo** of the field, and a **base point** that falls on the curve.
- The **order** of the base point, which is not independently selected but is a function of the other parameters, can be thought of graphically as the number of times the point can be added to itself until its slope is infinite, or a vertical line. The base point is selected such that the order is a large prime number.
- Bitcoin uses very large numbers for its base point, prime modulo, and order. In fact, all practical applications of ECDSA use enormous values. The security of the algorithm relies on these values being large, and therefore impractical to brute force or reverse engineer.

Case of Bitcoin

Elliptic curve equation: y^2 = x^3 + 7

Prime modulo = 2^256 – 2^32 – 2^9 – 2^8 – 2^7 – 2^6 – 2^4 – 1 = FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE FFFFFC2F

Base point = 04 79BE667E F9DCBBAC 55A06295 CE870B07 029BFCDB 2DCE28D9 59F2815B 16F81798 483ADA77 26A3C465 5DA4FBFC 0E1108A8 FD17B448 A6855419 9C47D08F FB10D4B8

Order = FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE BAAEDCE6 AF48A03B BFD25E8C D0364141

Who chose these numbers, and why? A great deal of research, and a fair amount of intrigue, surrounds the selection of appropriate parameters. After all, a large, seemingly random number could hide a backdoor method of reconstructing the private key. In brief, this particular realization goes by the name of secp256k1 and is part of a family of elliptic curve solutions over finite fields proposed for use in cryptography.

Private Keys and Public Keys

With these formalities out of the way, we are now in a position to understand private and public keys and how they are related. Here it is in a nutshell: In ECDSA,

- The private key is an unpredictably chosen number between 1 and the order.
- The public key is derived from the private key by scalar multiplication of the base point a number of times equal to the value of the private key. Expressed as an equation:

$$\text{public key} = \text{private key} * \text{base point}$$

This shows that the maximum possible number of private keys (and thus bitcoin addresses) is equal to the order.

Private Keys and Public Keys

In a continuous field we could plot the tangent line and pinpoint the public key on the graph, but there are some equations that accomplish the same thing in the context of finite fields. Point addition of *p* + *q* to find *r* is defined component-wise as follows:

c = (qy – py) / (qx – px)

rx = c^2 – px – qx

ry = c (px – rx) – py

And point doubling of *p* to find *r* is as follows:

c = (3px^2 + a) / 2py

rx = c^2 – 2px

ry = c (px – rx) – py

In practice, computation of the public key is broken down into a number of point doubling and point addition operations starting from the base point.

Private Keys and Public Keys

Let's run a back of the envelope example using small numbers, to get an intuition about how the keys are constructed and used in signing and verifying. The parameters we will use are:

Equation: y^2 = x^3 + 7 (which is to say, a = 0 and b = 7)

Prime Modulo: 67

Base Point: (2, 22)

Order: 79

Private key: 2

First, let's find the public key. Since we have selected the simplest possible private key with value = 2, it will require only a single point doubling operation from the base point. The calculation looks like this:

c = (3 * 2^2 + 0) / (2 * 22) mod 67

c = (3 * 4) / (44) mod 67

c = 12 / 44 mod 67

Private Keys and Public Keys

Here we have to pause for a bit of sleight-of-hand: how do we perform division in the context of a finite field, where the result must always be an integer? We have to multiply by the inverse, which space does not permit us to define here (Fermat's little theorem). In the case at hand, we have:

44^-1 = 32

Moving right along:

c = 12 * 32 mod 67

c = 384 mod 67

c = 49

$r_x = (49^2 - 2 * 2) \bmod 67$

$r_x = (2401 - 4) \bmod 67$

$r_x = 2397 \bmod 67$

$r_x = 52$

$r_y = (49 * (2 - 52) - 22) \bmod 67$

$r_y = (49 * (-50) - 22) \bmod 67$

$r_y = (-2450 - 22) \bmod 67$

$r_y = -2472 \bmod 67$

$r_y = 7$

Our public key thus corresponds to the point (52, 7). All that work for a private key of 2!

Private Keys and Public Keys

- This operation – going from private to public key – is computationally easy in comparison to trying to work backwards to deduce the private key from the public key, which while theoretically possible is computationally infeasible due to the large parameters used in actual elliptic cryptography.

  Therefore, going from the private key to the public key is by design a one-way trip.

- As with the private key, the public key is normally represented by a hexadecimal string.

- But wait, how do we get from a point on a plane, described by two numbers, to a single number?

- In an uncompressed public key the two 256-bit numbers representing the *x* and *y* coordinates are just stuck together in one long string. We can also take advantage of the symmetry of the elliptic curve to produce a compressed public key, by keeping just the *x* value and noting which half of the curve the point is on. From this partial information we can recover both coordinates.

Signing Data with the Private Key

Now that we have a private and public key pair, let's sign some data!

The data can be of any length. The usual first step is to hash the data to generate a number containing the same number of bits (256) as the order of the curve. Here, for the sake of simplicity, we'll skip the hashing step and just sign the raw data *z*. We'll call *G* the base point, *n* the order, and *d* the private key. The recipe for signing is as follows:

Choose some integer k between 1 and n – 1.

Calculate the point (x, y) = k * G, using scalar multiplication.

Find r = x mod n. If r = 0, return to step 1.

Find s = (z + r * d) / k mod n. If s = 0, return to step 1.

The signature is the pair (r, s)

Signing Data with the Private Key

As a reminder, in step 4, if the numbers result in a fraction (which in real life they almost always will), the numerator should be multiplied by the inverse of the denominator. In step 1, it is important that *k* not be repeated in different signatures and that it not be guessable by a third party. That is, *k* should either be random or generated by deterministic means that are kept secret from third parties. Otherwise it would be possible to extract the private key from step 4, since *s*, *z*, *r*, *k* and *n* are all known.

Let's pick our data to be the number 17, and follow the recipe. Our variables:

z = 17 (data)

G = (2, 22) (base point)

n = 79 (order)

d = 2 (private key)

Signing Data with the Private Key

Pick a random number:

k = rand(1, n – 1)

k = rand(1, 79 – 1)

k = 3 (is this really random? OK you got us, but it will make our example simpler!)

Calculate the point. This is done in the same manner as determining the public key, but for brevity let's omit the arithmetic for point addition and point doubling.

(x, y) = 3G

(x, y) = G + 2G

(x, y) = (2, 22) + (52, 7)

(x, y) = (62, 63)

x = 62

y = 63

Find *r*:

r = x mod n

r = 62 mod 79

r = 62

Find *s*:

s = (z + r * d) / k mod n

s = (17 + 62 * 2) / 3 mod 79

s = (17 + 124) / 3 mod 79

s = 141 / 3 mod 79

s = 47 mod 79

s = 47

Signing Data with the Private Key

Note that above we were able to divide by 3 since the result was an integer. In real-life cases we would use the inverse of *k*

$s = (z + r * d) / k \bmod n$

$s = (17 + 62 * 2) / 3 \bmod 79$

$s = (17 + 124) / 3 \bmod 79$

$s = 141 / 3 \bmod 79$

$s = 141 * 3^{-1} \bmod 79$

$s = 141 * 53 \bmod 79$

$s = 7473 \bmod 79$

$s = 47$

Our signature is the pair $(r, s) = (62, 47)$.

As with the private and public keys, this signature is normally represented by a hexadecimal string.

Verifying the Signature with the Public Key

We now have some data and a signature for that data. A third party who has our public key can receive our data and signature, and verify that we are the senders. Let's see how this works.

With *Q* being the public key and the other variables defined as before, the steps for verifying a signature are as follows:

Verify that r and s are between 1 and n – 1.

Calculate w = s-1 mod n

Calculate u = z * w mod n

Calculate v = r * w mod n

Calculate the point (x, y) = uG + vQ

Verify that r = x mod n. The signature is invalid if it is not.

Verifying the Signature with the Public Key

Why do these steps work? We are skipping the proof, but you can read the details here. Let's follow the recipe and see how it works.

Our variables, once again:

z = 17 (data)

(r, s) = (62, 47) (signature)

n = 79 (order)

G = (2, 22) (base point)

Q = (52, 7) (public key)

Verify that *r* and *s* are between 1 and *n* – 1. Check and check.

r: 1 <= 62 < 79

s: 1 <= 47 < 79

Verifying the Signature with the Public Key

Calculate *w*:

w = s^{-1} mod n

w = 47^{-1} mod 79

w = 37

Calculate *u*:

u = zw mod n

u = 17 * 37 mod 79

u = 629 mod 79

u = 76

Calculate *v*:

v = rw mod n

v = 62 * 37 mod 79

v = 2294 mod 79

v = 3

Calculate the point (*x*, *y*): (x, y) = uG + vQ

Let's break down the point doubling and addition in *uG* and *vQ* separately.

uG = 76G

uG = 2(38G)

uG = 2(2(19G))

uG = 2(2(G + 18G))

uG = 2(2(G + 2(9G)))

uG = 2(2(G + 2(G + 8G)))

uG = 2(2(G + 2(G + 2(4G))))

uG = 2(2(G + 2(G + 2(2(2G)))))

uG = 2(2(G + 2(G + 2(2(2(2, 22))))))

uG = 2(2(G + 2(G + 2(2(52, 7)))))

uG = 2(2(G + 2(G + 2(25, 17))))

uG = 2(2(G + 2((2, 22) + (21, 42))))

uG = 2(2(G + 2(13, 44)))

uG = 2(2((2, 22) + (66, 26)))

uG = 2(2(38, 26))

uG = 2(27, 40)

uG = (62, 4)

Verifying the Signature with the Public Key

And now for *vQ*:

vQ = 3Q

vQ = Q + 2Q

vQ = Q + 2(52, 7)

vQ = (52, 7) + (25, 17)

vQ = (11, 20)

Putting them together:

(x, y) = uG + vQ

(x, y) = (62, 4) + (11, 20)

(x, y) = (62, 63)

Clearly step 5 is the bulk of the work. For the final step,

Verify that r = x mod n

r = x mod n

62 = 62 mod 79

62 = 62

Our signature is valid!

Conclusion

We have developed some intuition about the deep mathematical relationship that exists between public and private keys. We have seen how even in the simplest examples the math behind signatures and verification quickly gets complicated, and we can appreciate the enormous complexity which must be involved when the parameters involved are 256-bit numbers. We have seen how the clever application of the simplest mathematical procedures can create the one-way “trapdoor” functions necessary to preserve the information asymmetry which defines ownership of a bitcoin. And we have newfound confidence in the robustness of the system, provided that we carefully safeguard the knowledge of our private keys.

In other words, this is why it is commonly said that bitcoin is “backed by math”.

Academic

CS 251: Bitcoin and Cryptocurrencies

Course syllabus and readings

Fall 2020

Every lecture is accompanied by readings that support and expand on what was covered in the lecture. In the listings below we use NBFMG to refer to the course textbook *Bitcoin and Cryptocurrency Technologies: A Comprehensive Introduction*.

Lecture 1: Mon 9/14/20 (DB)

Intro to cryptography & cryptocurrencies [pdf, pptx]

References:
- NBFMG Chapter 1

Bitcoin mechanics

Lecture 2: Wed 9/16/20 (DB)

Bitcoin nuts and bolts [pdf, pptx]

References:
- NBFMG Chapter 3
- Bitcoin Developer Guide sections: Block Chain, Transactions, P2P Network
- Bitcoin: A Peer-to-Peer Electronic Cash System, by Satoshi Nakamoto

Lecture 3: Mon 9/21/20 (DB)

Wallets: managing and protecting crypto assets [pdf, pptx]

References:
- NBFMG Chapter 4
- Bitcoin Developer Guide section: Wallets

Consensus protocols

Lecture 4: Wed 9/23/20 (BB)

Consensus: network models, corruption tolerance, sybil resistance [pdf, pptx]

References:
- Foundations of Consensus Chapter 3, 6 and 7
- Rethinking Large Scale Consensus

Lecture 5: Mon 9/28/20 (BB)

Nakamoto Consensus: security, attacks and incentives [pdf, pptx]

References:
- Analysis of the Blockchain Protocol in Asynchronous Networks

https://cs251.stanford.edu/syllabus.html

Innovate with Math