In the name of God

Hausdorff Distance

A Note on the Hausdorff
Distance Between Norm
Balls and Their Linear Maps

Introduction

As a fundamental tool in geometry and data analysis, the Hausdorff distance is a distance metric for the difference between two sets. This concept is used to compare forms, analyze the similarity of patterns, and study the effect of changes in the use of forms. Its applications include image processing to compare different forms, in detecting and identifying patterns in statistical data, and in engineering sciences to investigate stability and safety.

The purpose of the article

The purpose of this paper is to accurately calculate the Hausdorff distance between different norm balls, especially l_1,l_2
in Euclidean spaces with limited dimensions. The paper also examines the effect of linear transformations on these distances and offers suggestions for practical applications of these findings in control systems.

Report path

- Definition of basic concepts
- some results of the paper
- Support functions and Control theory
- Set-valued integral
- Estimates and numerical limits
- Linear transformations
- Conclusion and summery

Definition of basic concepts

Norms:

Euclidean norm(l_2)

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Norm l_1

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

Norm l_∞

(x is a vector and x_i are its components.)

Definition of basic concepts

- Norm balls:
- Consider 1<p<∞ for v=(x,y)∈ $\mathbb{R} \times \mathbb{R}$:

$$\| v \|_p = \| (x, y) \|_p = (x^p + y^p)^{\left(\frac{1}{p}\right)}$$

- We define the p-norm ball as the set of all vectors in $\mathbb{R} \times \mathbb{R}$ such that de p-distance to the origin is less than 1.

$$B_p = \left\{ v \in \mathbb{R}^2 : \| v \|_p \leq 1 \right\}$$

We call B_p a p-norm. [1]

1. Visit this address for understanding more: https://www.geogebra.org/m/pyxfvyyk

Definition of basic concepts

Hausdorff distance:

Given compact $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$, the two sided Hausdorff distance δ between them is a mapping $\delta : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}_{\geq 0}$ defined as

$$\delta(\mathcal{X}, \mathcal{Y}) := \max \left\{ \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \| \boldsymbol{x} - \boldsymbol{y} \|_2 , \ \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \| \boldsymbol{x} - \boldsymbol{y} \|_2 \right\}, \tag{1}$$

And as an example of Hausdorff distance: to compute Hausdorff distance of two arbitrary set of *A* (a circle with center of (0,0) and radius 1) and set of *B* (a circle with center of (2,0) and radius 1).

Example of computing a Hausdorff distance

- **Definition of points and distances:**
- point $a \in A$ On the circle border of A is expressed: $(\cos\theta\,,\ \sin\theta)$
- point $b \in B$ On the circle border of B is expressed: $(2 + \cos\varphi\,,\ \sin\varphi)$
- **Calculating the final distance:** These distances show that the largest distance from the closest point of one circle to another is equal to 2. Therefore, the Hausdorff distance between these two circles is equal to 2 units.
- See the Separation axiom to get what Hausdorff space is.

Definition of basic concepts

- **Dual of norms:**

Dual norm $\|\cdot\|$ on a vector space V, is corresponding norm on dual space V^* if $\varphi \epsilon V^*$ be a continuous linear function, dual norm $\|\cdot\|$ which usually represents by $\|\cdot\|^*$ is defined as follows: $\|\varphi\|^* = sup_{\|x\|\leq 1}|\varphi(x)|$

That is, the largest value that ϕ can take on a single vector in V.

As a specific example, consider norm p and its dual

Suppose V=$\mathbb{R}^n$and $\|.\|_1$norm is l_1 (or Manhattan norm) on this vector space: $\|x\|_1 = \sum_{i=1}^{n}|x_i|$

The dual of this norm, l_∞(or infinitely norm) will be: $\|\varphi\|_\infty = \max_{1\leq i\leq n}|\varphi_i|$

explanation

For a better understanding, if the norm 1-norm of a vector in space is equal to the sum of the absolute values of its components, its dual, i.e. infinite norm, will be equal to the maximum value of these components.

Definition of basic concepts

- **Convexity:**

A set *C* in a normal space is a convex set if for any two points x,y in *C*, any point that lies along the line between them is also in *C.* Mathematically:

$$\forall x, y \in C, \forall \lambda \in [0,1], \quad \lambda x + (1-\lambda) y \in C$$

Definition of basic concepts

- **Holder conjugate:**

  Consider Holder conjugate exponents p and q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Multiplying through by pq gives $p + q = pq$. So conjugate exponenets are just those real numbers whose product and sum are the same. I have two questions.

- **Pseudo-inverse:**

$$A^{\dagger} = A^T(AA^T)^{-1}$$

In which A is a matrix and A^T is its transpose.

- **Contour plot:**

It's a graphical technique for representing a 3 dimensional surface by plotting constant z slices, called contours, on a 2 dimensional format.

Definition of basic concepts

- **Minkowski sum:**
- In geometry, the Minkowski sum of two sets of position vectors *A* and *B* in Euclidean space is formed by adding each vector in *A* to each vector in *B*:

$$A + B = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A,\ \mathbf{b} \in B\}$$

- **Reach sets:**
- Reach Sets (reachable sets) are a fundamental concept in control theory, dynamical systems, and computer science. Simply put, a Reach Set is the set of all possible states that a system can reach within a given time interval, subject to the system's dynamics and constraints. (see more in “Verification of Safety Critical Systems” book by G. J. Pappas & S. Sastry)

Definition of basic concepts

- **Bounded and closed and compact sets**
- A set A in a topological space X is known as a **closed set** if all limit points of A also exist in A itself. Equivalently, a set A can be defined as closed if its complement, X\A i.e., is an open set.
- A set A in the space of real numbers or in any vector space is known as a **bounded set** if it is

possible to find a real number M such that for each element x in A the following relationship holds: $||x|| \leq M$

(||. || Here usually refers to the norm of the elements, which can be Euclidean norm or any other norm.)

In other words, set A must be inside a ball with origin center and radius M

- A set K in a topological space is said to be **compact** if for every open cover of the set K one

can choose a finite subcover of that cover that still covers K.

Propositions and theorem

Holder inequality

$$\sum_{i=1}^{d} |a_i b_i| \le \left(\sum_{i=1}^{d} |a_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^{d} |b_i|^{\frac{r}{r-1}} \right)^{1-\frac{1}{r}} \qquad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d, \quad 1 \le r \le \infty,$$

- While r and r/r-1 are Holder conjugates.

Triangle inequality also is inferred by this.

Now some results of the paper

- essay uses $[\![n]\!] := \{1,2,\ldots,n\}$ to denote the set of natural numbers from 1 to n.
- And $\lfloor k \rfloor$ stands for the floor function that returns the greatest integer less than or equal to its real argument.

(***D*-norm**) For $1 \leq k \leq d$, the D-norm of $\boldsymbol{x} \in \mathbb{R}^d$ is

$$\|\boldsymbol{x}\|_k^D := \max_{\{S \cup \{t\} | S \subseteq [\![d]\!], \mathrm{card}(S) \leq \lfloor k \rfloor, t \in [\![d]\!] \setminus S\}} \left\{ \sum_{i \in S} |x_i| + (k - \lfloor k \rfloor)|x_t| \right\}. \qquad (2)$$

- Now paper introduces a now norm here in which d is the radius and For k=1, the norm reduces to the l_∞norm, i.e.,$\|x\|_1^D = \|x\|_\infty$. For k=d, the norm reduces to the l_1 norm, i.e., $\|x\|_d^D = \|x\|_1$. For 1<k<d, the norm can be thought of as a polyhedral interpolation between the l_∞ and the l_1 norms

some results of the paper

- **There is another norm that is not our goal to introduce**

(k **largest magnitude norm**) For $k \in [\![d]\!]$, the k largest magnitude norm of $\boldsymbol{x} \in \mathbb{R}^d$ is

$$\|\boldsymbol{x}\|_{[k]} := |x_{i_1}| + |x_{i_2}| + \ldots + |x_{i_k}|,$$

some results of the paper

- We consider the case when in (3), the sets $\mathcal{K}_1 \equiv \mathbb{B}^d_{p_1}$, $\mathcal{K}_2 \equiv \mathbb{B}^d_{p_2}$, the unit l_{p_1} and l_{p_2} norm balls in $\mathbb{R}^d$, d≥2, for 1≤ $p_1 \leq p_2 \leq \infty$. Clearly, the Hausdorff distance δ =0 for $p_1 = p_2$, and δ >0 otherwise. for 1≤ $q_1 \leq q_2 \leq \infty$. By monotonicity of the norm function, we know that $\|.\|_{q_1} \leq \|.\|_{q_2}$. Therefore, the Hausdorff distance in this case becomes
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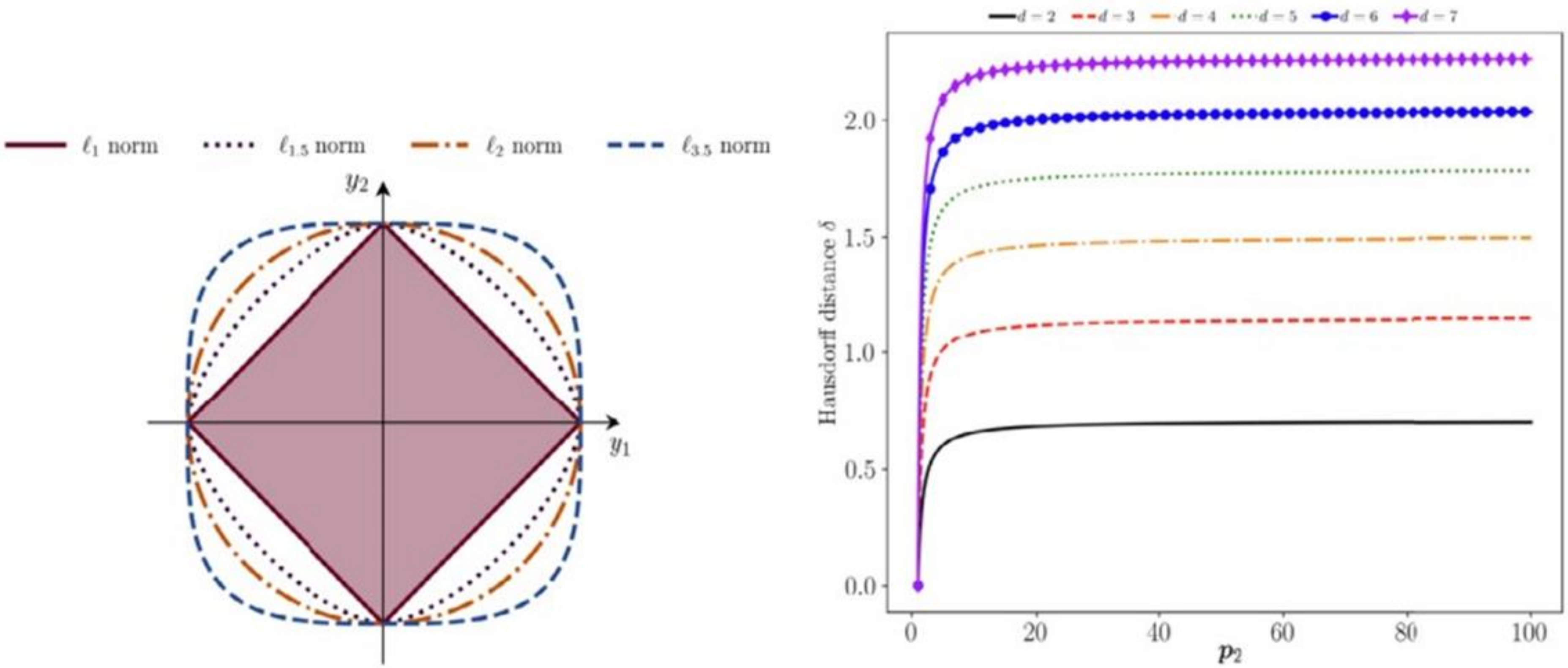
$$\delta\left(\mathcal{K}_1, \mathcal{K}_2\right) = \delta\left(\mathbb{B}^d_{p_1}, \mathbb{B}^d_{p_2}\right) = \sup_{\|\boldsymbol{y}\|_2 = 1} \left(\|\boldsymbol{y}\|_{q_2} - \|\boldsymbol{y}\|_{q_1}\right), \qquad (3)$$

some results of the paper

- For $1 \leq p_1 \leq p_2 \leq \infty$, we have

$$\delta\left(\mathbb{B}_{p_1}^d, \mathbb{B}_{p_2}^d\right) = d^{-\frac{1}{2}}\left(d^{\frac{1}{q_2}} - d^{\frac{1}{q_1}}\right),$$

- where q_i denotes the Hölder conjugate of p_i for $i \epsilon \{1,2\}$.
- As the intuition suggests, for a fixed p_1, larger p_2 results in a larger δ in a given dimension d≥2. see fig1 on the next slide



(a) Convex ℓ_p norm balls in $\mathbb{R}^2$

(b) The Hausdorff distance (9) for fixed $p_1 = 1$

Fig. 1 Understanding the Hausdorff distance δ between the unit ℓ_{p_1} and ℓ_{p_2} norm balls in $\mathbb{R}^d$, $d \geq 2$, for $1 \leq p_1 < p_2 \leq \infty$

some results of the paper

now next a consideration of a generalized version of (3) given by

$$\delta(\mathcal{K}_1, \mathcal{K}_2) = \sup_{\|\boldsymbol{y}\|_2 = 1} \left(\|\boldsymbol{T}\boldsymbol{y}\|_{q_2} - \|\boldsymbol{T}\boldsymbol{y}\|_{q_1} \right), \qquad 1 \leq q_2 < q_1 \leq \infty,$$

where the matrix $T \epsilon \mathbb{R}^{m \times d}$,$m \leq d$, has full row rank m.
This formula computes the Hausdorff distance between two compact convex sets in $\mathbb{R}^d$ obtained as the linear transformations of the m-dimensional l_{p_1} and l_{p_2} unit norm balls via $T^T \epsilon \mathbb{R}^{d \times m}$

$$\mathcal{K}_1 \equiv \boldsymbol{T}^\top \mathbb{B}^d_{p_1}, \quad \mathcal{K}_2 \equiv \boldsymbol{T}^\top \mathbb{B}^d_{p_2}.$$

some results of the paper

We can also consider the Hausdorff distance between the common linear transforms of different polyhedral norm balls equals:

$$\delta\left(\boldsymbol{T}\mathcal{K}_1, \boldsymbol{T}\mathcal{K}_2\right) = \left(\frac{1}{k_1} - \frac{1}{k_2}\right) \underbrace{\sup_{\|\boldsymbol{y}\|_2=1} \|\boldsymbol{T}^\top \boldsymbol{y}\|_1}_{=:\|\boldsymbol{T}^\top\|_{2\to1}} = \left(\frac{1}{k_1} - \frac{1}{k_2}\right) \|\boldsymbol{T}\|_{\infty\to2}, \qquad (4)$$

We have to clarify that what does $\|T\|_{\infty\to2}$ indicate? To ascertain this we have:

For $1 \le p, q \le \infty$, matrix $M \epsilon \mathbb{R}^{m\times n}$ viewed as a linear map $M :\ l_p(\mathbb{R}^n) \to l_q(\mathbb{R}^n)$, has an associated induced operator norm

$$\|\boldsymbol{M}\|_{p\to q} := \sup_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|\boldsymbol{M}\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p} = \sup_{\|\boldsymbol{x}\|_p=1} \|\boldsymbol{M}\boldsymbol{x}\|_q,$$

some results of the paper

- As the result of concepts on the previous slide, the operator norm $\|T\|_{2\to q_1}$ is known to be equal to the optimal value of the following convex optimization problem:

$$\mathrm{OPT} := \max_{\boldsymbol{X} \succeq \mathbf{0}} \sqrt{\| \operatorname{dg}\left(\boldsymbol{T}\boldsymbol{X}\boldsymbol{T}^{\top}\right) \|_{\frac{q_1}{2}}}$$

$$\text{subject to } \| \operatorname{dg}(\boldsymbol{X}) \|_1 \leq 1,$$

- where dg(.) takes a square matrix as its argument and returns the vector comprising of the diagonal entries of that matrix.

Integral version

- To compute hausdorff distance between the two d-dimensional compact convex sets bellow:

$$\mathcal{K}_1 \equiv \int_0^t \boldsymbol{T}^\top(\tau)\mathcal{P}_1 \mathrm{d}\tau$$

$$\mathcal{K}_2 \equiv \int_0^t \boldsymbol{T}^\top(\tau)\mathcal{P}_2 \mathrm{d}\tau$$

- It is enough to consider a further generalization of (3) for it.

$$\delta(\mathcal{K}_1, \mathcal{K}_2) = \sup_{\|\boldsymbol{y}\|_2 = 1} \int_0^t \left(\|\boldsymbol{T}(\tau)\boldsymbol{y}\|_{q_2} - \|\boldsymbol{T}(\tau)\boldsymbol{y}\|_{q_1} \right) \mathrm{d}\tau, \qquad 1 \leq q_2 < q_1 \leq \infty,$$

- where for each $\tau \in [0, t]$, the matrix $\boldsymbol{T}(\tau) \in \mathbb{R}^{m \times d}, m \leq d,$ is smooth in and $\boldsymbol{\tau}$ has full row rank m.

some results of the paper

Now we bring this Prop. without its proof to get one of essay's conclusions

Proposition 3 (***Upper bound***) *Let* $\boldsymbol{T} \in \mathbb{R}^{m \times d}$. *Then for* $1 \le q_2 < q_1 \le \infty$, *we have*

$$\sup_{\|\boldsymbol{y}\|_2=1} \left(\|\boldsymbol{T}\boldsymbol{y}\|_{q_2} - \|\boldsymbol{T}\boldsymbol{y}\|_{q_1}\right) \le \left(m^{\frac{1}{q_2}-\frac{1}{q_1}} - 1\right) \|\boldsymbol{T}\|_{2\to q_1}. \qquad (5)$$

Notice that in this case, (5) directly yields

$$\delta\left(\mathcal{K}_1, \mathcal{K}_2\right) \le \left(m^{\frac{1}{q_2}-\frac{1}{q_1}} - 1\right) \int_0^t \|\boldsymbol{T}(\tau)\|_{2\to q_1} \,\mathrm{d}\tau.$$

Support functions

Support function:

- The support function for the convex set *C* in the direction of u is defined as:

$$h_C(u) = \sup_{x \in C} \langle u, x \rangle$$

- As an example for two finite set *C*= {(x,y)| $x^2 + y^2 \leq 1$} in $\mathbb{R}^2$and a vector u=(1,1) we have:
- h_C(u)= $\sup_{x \in C} \langle x, u \rangle$= $\sup_{x \in C} \left\langle \binom{x}{y}, \binom{1}{0} \right\rangle$= $\sup_{x \in C}$ x=1
- The support function represents the maximum limit of expansion of the set in a certain direction and helps to provide the geometric properties of convex sets. This concept helps to understand how to represent a set in Euclidean space and is used in calculating different distances including Hausdorff distance.

Support functions

- **Applications**

In optimization problems, the definition and application of the support function helps to simplify and solve problems focused on convex sets and their more complete evaluation. This function is directly important in calculating distances between geometric sets and is a key tool for complex geometric analysis and measuring distances in the form of geometric sets.

Calculating the Hausdorff distance leads to the solution of non-convex optimization problems that reveal complex landscapes with multiple local maxima and minima.

One of the practical applications of these concepts can be mentioned in control theory, especially in evaluating the safety of controlled dynamic systems that are modeled with input uncertainty.

control theory

- **Applications in control theory**

The theoretical results in the article are used in the field of control systems, especially in the calculation of reach sets in the presence of uncertainties. Changes in these sets by norm balls can help to better evaluate the stability conditions of the systems.

- **Practical importance**

This knowledge can practically help to design robust and stable control systems under conditions of uncertainty in inputs and calculate probabilistic sessions for planning the route and control points in adverse conditions. Practical examples include guided systems such as autonomous vehicles and airplanes.

Estimates and numerical limits

To confirm the theoretical results, various numerical experiments have been performed. These experiments evaluate the changes of the Hausdorff distance in different conditions and show the practical applications of the proposed theories.

In this section, challenges such as accuracy in calculations and stability in the face of high-dimensional space are raised. They present techniques to overcome these challenges, improving the accuracy of the estimates and the behavior of the objective functions.

Linear transformations

Examining the effect of linear transformations A on the distribution of norm balls and evaluating the change in Hausdorff distance shows that linear transformation can have a great impact on the shape and distance of balls. This transformation using the formula bellow evaluated and allows us to understand the effects and geometric changes.

$$d_H(A(B_1), A(B_2)) = \sup_{\|x\|_2=1} \|Ax\|_1$$

The paper also examines the effect of arbitrary or random linear transformations on the Hausdorff distance and sets bounds for the expected Hausdorff distance while the applied transformations have random values with zero mean.

Linear transformations

- **Upper bounds for distances**

This section presents upper bounds for the Hausdorff distance (which depend on the properties of the linear map as a soft operator) under linear transformations.

These bounds help to analyze stability and better understand the internal order of systems. The authors also investigate how the random features of the transformation matrices affect (we know that any linear transformation can be shown as a matrix).

This article, beside investigating the effect of arbitrary or random linear transformations on the Hausdorff distance, for the expected Hausdorff distance while the applied transformations have values are random with zero mean, it sets boundaries. So, for example, if A is a random linear transformation matrix, it can be shown that the Hausdorff distance will not exceed a certain value.

Limitations of existing methods:

The limitations of current methods for general linear mappings are discussed and the need for better methods for calculating the Hausdorff distance is discussed.

Suggestions for future research

- Suggestions for developing more optimal algorithms for calculating the Hausdorff distance and its applications in control theory.

Summary

In fact, the authors of the paper have tried to calculate the Hausdorff distance, which is a type of measurement between two sets, between soft spheres under the influence of different linear mappings. Then use these calculations in the control and analysis of dynamic systems.

Resources

Beside all resources named in the essay, there are some that I have used to make it easier for me to understand and read. And here are them:

- A Note on the Hausdorff Distance Between Norm Balls and Their Linear Maps by Shadi Haddad Abhishek Halder Received: 3 August 2022 / Accepted: 27 July 2023

  Set-Valued and Variational Analysis (2023) 31:30

  https://doi.org/10.1007/s11228-023-00692-1
- wikipedia.org
- archive.org (to read some books)
- Book
- geogebra.org

Thank you

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The end