NILPOTENCY IN MATH.

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Abstract. In this lecture we state some but important applications of the concept "**nilpotency**" in math.

Let R be an associative ring that means:

$$(R,+) \text{ is an additive group, and}$$
$$\forall a,b,c \in R,\ a(bc)=(ab)c,\ a(b+c)=ab+ac, (b+c)a=ba+ca.$$

Definition 0.1. *An element $a \in R$ is called* **nilpotent** *if $\exists n \geq 1$, such that $a^n = 0$.*

The set of all nilpotent elements of R is denoted by $\mathbf{nil}(R)$.

Example 0.2. *If $A=(a_{ij})_{n\times n}$ such that $a_{ij}=0$ for $i \geq j$ then $A^n=0$.*

Definition 0.3. *Let $\emptyset \neq X \subseteq R$. Then X is called;*

1) **right ideal (left ideal)***: $X \leq (R,+)$ and $XR \subseteq X$ ($RX \subseteq X$), and* **ideal** *if it is both right and left ideal (write $X \trianglelefteq R$).*

2) **nilpotent***: $\exists n \geq 1$ $X^n = 0$, or equivalently, $\exists n \geq 1$, $\forall x_1, \cdots, x_n \in X$ we have $x_1x_2\cdots x_n = 0$.*

3) **(right) T-nilpotent***: $\forall\ x_1, x_2, \cdots \in X$ $\exists n \geq 1$ such that $x_nx_{n-1}\cdots x_1 = 0$. ("T" for transfinite)*

4) **nil***: $\forall x \in X$, $\exists n_x \geq 1$ such that $x^{n_x} = 0$, and nil(R) = the set of all nilpotent elements of R.*

$$\text{nilpotent} \Rightarrow \text{T-nilpotent} \Rightarrow \text{nil}$$

Lemma 0.4. *If $x, y \in nil(R)$ and $xy = yx$ then $xy, x + y \in nil(R)$. Furthermore, if R is commutative then for all $I \trianglelefteq R$, $\sqrt{I} = \{x \in R \mid \exists n_x \geq 1,\ x^{n_x} \in I\}$. Hence, $nil(R/I) = \sqrt{I}/I$ and $nil(R) = \sqrt{(0)}$.*

Example 0.5. *1) If $R = \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3} \times \cdots$ then $nil(R)$ is a nil ideal but the index of nilpotency in R has not bounded.*

2) For $n \geq 2$, $\mathbb{Z}_n \simeq \mathbb{Z}/ < n >$. $\text{nil}(\mathbb{Z}_n) \ \leftrightarrow \ \{x \mid \exists k \geq 1,\ n|x^k\}$.

Example 0.6. *If $R = \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3} \times \cdots$ then $nil(R)$ is a nil ideal but the index of nilpotency in R has not bounded.*

Köthe Conjecture: *R has no nonzero nil ideal $\Rightarrow R$ has no nonzero nil one sided ideal.*

Note: *I is a nil one sided ideal of R $\Rightarrow I \subseteq J(R)$.* $J(R)$ = the intersection of all maximal right(left) ideals of R.

J. Levitzki (1939) unpublished, (1950) published, proved that *if right ideals in R are finitely generated then Köthe conjecture is true.*

Theorem 0.7. *The following statements are equivalent to the Köthe conjecture.*
(1) The sum of two nil right (left) ideal is a nil right (left) ideal.
(2) If I is a nil ideal of R then so is the ideal $M_n(I)$ in the ring $M_n(R)$ for all $n \geq 1$.
(3) If I is a nil ideal of R then so is the ideal $I[x] \subseteq J(R[x]$.

Algebraic Geometry Points:

K is field, $R = K[x_1, ..., x_n]$ and $V = K \times \cdots \times K = K^{(n)}$. $\forall f = f(x_1, ..., x_n) \in R$ and $p = (a_1, ..., a_n) \in V$, we write $f.p = f(a_1, ..., a_n)$.

Let $\forall I \trianglelefteq R$, $V(I) = \{p \in V \mid Ip = 0\}$ and $\forall U \subseteq V$, $I(U) = \{f \in R \mid fU = 0\}$. Then we have:

$$\left\{I \trianglelefteq R \mid \sqrt{I} = I\right\} \longleftrightarrow \{I \trianglelefteq R \mid nil(R/I) = 0\} \longleftrightarrow \{V(I) \mid I \trianglelefteq R\} \subseteq V.$$

The set of $\{V(I)\ I \trianglelefteq R\}$ form the closed subsets of a *topology* on V, named " Zariski topology".

If $R = F[x]$, $f(x) := f \in R$, $I =< f >= fR$. Consider $f = P_1^{\alpha_1} \cdots P_t^{\alpha_t}$ where P_i is a prime element in $F[x]$, then $\mathrm{nil}(R/I) = 0 \Leftrightarrow \alpha_i = 1\ \forall i$.

If $A \in Mat_{n\times n}(F)$, then $\exists 0 \neq f(x) \in F[x]$ such that $F[A] \simeq F[x]/ < f(x) >$ where $f(A) = 0$ with minimum degree. Thus if F is algebraically closed, then $\mathrm{nil}(R/I) = 0 \Leftrightarrow A = P^{-1}DP$ for some diagonal matrix D.

Derivation algebra Points:

$D : R \to R$ is called a **derivation** if $\forall x, y \in R$, $\quad D(x + y) = D(x) + D(y)$ and $D(xy) = xD(y) + D(y)x$.

Example 0.8. *1) If K is field, $R = K[x_1, ..., x_n]$ and $D = \partial/\partial x_i$.*

2) For any ring R and $x \in R$, define $ad_x : R \to R$ with $ad_x(y) = xy - yx$. ad_x is called **inner derivation** *on R.*

Note that if $D = \partial/\partial x_i$ and degree_{x_i} of $f < m$ then $D^m(f) = 0$. Thus if $x_i = y$ and $f = \sum_n a_n y^n$ then we can compute $a_k = D^k(0)/k!$.

Charles Lanski and W.S. Martindale et al. Study derivations D on R such that there are extension $R \leq Q$ and $q \in Q$ with $D(x) = qx - xq\ \forall x \in R$. Since

$$(ad_q)^n(x) = \sum_{i+j=n} (-1)^j \binom{n}{j} q^i x q^j$$

hence if $q^n = 0$ then $(ad_q)^n = 0$. Martindale showed that if $(ad_q)^n = 0$ then $\exists\ \lambda \in Cent(Q)$ such that $q - \lambda$ is a nilpotent element. Note $ad_q = ad_{q-\lambda}$.

Homological points :

Definition 0.9. *Let R be a ring. An abelian group $(M,+)$ is called right R-module. If there is action $M \times R :\to M$ such that*

$$\forall a, b \in R \text{ and } x, y \in M,\ (xa)b = x(ab),\ x(a+b) = xa + xb,\ (x+y)a = xa + ya$$

.

Theorem 0.10 (H. Bass 1960)**.** *The following statements are equivalent for a ring R.*
(1) Every right R-module has a projective cover.
(2) $R/J(R)$ isomorphic to a finite direct product of matrices over division rings and $J(R)$ is a right T-nilpotent.

If P is a projective R-module (i.e., $\exists Q,\ P \times Q \simeq R^{(\Lambda)}$) and $\exists f : P \to M$ such that $Ker(f)$ is small in P (i.e., $(Ker(f) + N = P) \Rightarrow N = P$), then P is called a "*projective cover*" for M.

$\cdots \to P_1 \xrightarrow{f_1} Ker(f) \subseteq P_0 = P \xrightarrow{f} M$. Then we obtain a projective resolution for M,

$$\cdots P_n \to \cdots \to P_1 \to P_0 = P \to M$$

Fine rings, T.Y. Lam (2016). R is called *fine* if

$$\forall x \in R, \exists u \in U(R), a \in nil(R), \quad x = u + a$$

Dimension and Torsion Theory points :

$$dim(M) \leq dim(N) + dim(M/N) \ \ and \ \ dim(R) = Sup_M\{dim(M)\}$$

$$N, M/N \in C \Rightarrow M \in C$$

For any $I \trianglelefteq R$ with $MI = 0$, we have $L(M_R) = L(M_{R/I})$. Now if $I^n = 0$ we have:

$$M \supseteq MI \supseteq MI^2 \supseteq \cdots \supseteq MI^n = 0$$

and all MI^i/MI^{i+1} are R/I-module. In commutative case R.
-$dim(R) = Sup_P\{dim(R_P)\}$ for every prime ideal P. Here, $R_P = \{a/b \mid a \in R, b \notin P\}$.
- for every prime ideal P, $\Rightarrow R_P$ is a ring with unique maximal ideal.
- R is a ring with maximal ideal $I \Rightarrow R/I^n$ is a ring with nilpotent maximal ideal $(n \geq 1)$.
- R is a ring with maximal ideal I such that $I^n = 0 \Rightarrow R/I = F$ is a field and all MI^i/MI^{i+1} are vector space on F.

T-nilpotent ideals:

Definition 0.11. *A nonzero R-module M is said to be* **weak-generator** *whenever*

$$Hom_R(M, X) = 0 \Rightarrow X = 0$$

For any R-module M, let $ann_R(M) = \{x \in R \mid Mx = 0\}$.

Theorem 0.12 (Smith-Vedadi 2006)**.** *Let R be a ring Morita equivalent to commutative ring.*
(i) A finitely generated R-module M is weak-generator $\Longleftrightarrow$ $ann_R(M)$ is T-nilpotent.

(ii) An ideal I is T-nilpotent $\Leftrightarrow$ the ideal $I[x]$ is T-nilpotent $\Leftrightarrow$ the ideal $Mat_n(I)$ is T-nilpotent ($n \geq 1$).

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