Addition Cayley graphs and some of their properties

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August 19, 2015

Doctor of Philosophy Thesis

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Abstract

Let S be a subset of a finite abelian group G. The Cayley sum graph $\mathrm{Cay}^+(G,S)$ of G with respect to S is a graph whose vertex set is G and two vertices g and h are joined by an edge if and only if $g+h \in S$. We say that G is a Cayley sum integral (resp. k-integral) group if for all $S \subseteq G$ (resp. $S \subseteq G$ with $|S| = k$), the graph $\mathrm{Cay}^+(G,S)$ has integer eigenvalues.

In this thesis we show that a group is an integral Cayley sum abelian group if and only if it is a cyclic group of order 3 or an elementary abelian 2-group. Also, we show that every elementary abelian 2-group and elementary abelian 3-group are the only 2-integral Cayley sum abelian groups, and every 3-integral Cayley sum abelian group is represented by $\mathbb{Z}_2^n \oplus \mathbb{Z}_3^m$ or $\mathbb{Z}_2^n \oplus \mathbb{Z}_4^m$, where $n \geq 0$ and $m \in \{0,1\}$. As a result, we prove that there are exactly six simple connected cubic integral Cayley sum graphs. In particular, if G is a finite arbitrary group, then for some square free subset S of size 3, $\mathrm{Cay}^+(G,S)$ is connected and integral if and only if G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^2,\ \mathbb{Z}_6,\ D_6,\ \mathbb{Z}_8,\ \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ \mathbb{Z}_2^3,\ \mathbb{Z}_{12},\ \mathbb{Z}_2 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_{12},\ \mathbb{Z}_4 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

For every $i, j \in \mathbb{Z}_n$, a matrix $A = [a_{ij}]$ is anti-circulant if $a_{ij} = a_{i+1,j-1}$. A graph is called anti-circulant if it has an anti-circulant adjacency matrix. In this research, by proving that a graph is anti-circulant if and only if it is a Cayley sum graph of a cyclic group, we obtain some results on the isomorphisms, connectivity and vertex transitivity about anti-circulant graphs. We also compute the spectrum of real anti-circulant matrices.

In addition, by working on the relationship between the algebraic properties of groups and properties of graphs, we study the parameters such as diameter, girth, (total) domination number and (total) dominating set and some other important properties of some classes of Cayley sum graphs.

Key Words: Cayley sum graph; Cayley sum color graph; anti-circulant graph; integral graph; integral Cayley sum graph; (total) domination number; (total) dominating set.

MSC 2010: 20K01, 05C25, 05C50, 05C45, 15A18.

Introduction

Let G be a finite group and $\alpha : G \to \mathbb{R}$ be a real-valued function on G. The Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$ is the complete directed graph with vertex set G where each arc $(x, y) \in G \times G$ is associated with the color $\alpha(xy)$. If α is the characteristic function of a subset $S \subseteq G$, that is,

$$\alpha(g) = \begin{cases} 1 & \text{if} \quad g \in S \\ 0 & \text{if} \quad g \notin S, \end{cases}$$

then we use $\mathrm{Cay}^+(G, S)$ instead of $\mathrm{Cay}^+(G, \alpha)$ and call it the Cayley sum digraph of G with respect to S. We regard $\mathrm{Cay}^+(G, S)$ as a digraph formed by deleting the arcs of color 0 from the color graph.

The adjacency matrix A of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$ is a $|G| \times |G|$ matrix whose row-g column-h entry a_{gh} is $\alpha(gh)$ for all $g, h \in G$. In a Cayley sum (di)graph $\mathrm{Cay}^+(G, S)$, there is an (arc) edge from g to h if and only if $a_{gh} = \alpha(gh) = 1$, equivalently $h = g^{-1}s$, for some $s \in S$.

Note that if S is a normal subset of G, i.e. for every $g \in G, g^{-1}Sg = S$, then $\mathrm{Cay}^+(G, S)$ is an undirected graph. Also, if there exists $g \in G$ such that $g^2 \in S$, then the edge $\{g, g\}$ is a semi-edge; a semi-edge is an edge with one endpoint. Unlike a loop, a semi-edge contributes just one to both the valency of its endpoint and the corresponding diagonal entry of the adjacency matrix. With this convention, $\mathrm{Cay}^+(G, S)$ is a regular graph of degree $|S|$, as the set of adjacent vertices of g is $\{g^{-1}s \mid s \in S\}$. In the case that G is a finite abelian group, Cayley sum graphs are known in the literature under names *addition Cayley graphs* [13, 16, 17], *addition graphs* [9] and *sum graphs* [10].

The study of Cayley sum graphs has been the object of some papers, for example, independence number [2], hamiltonicity [9, 16]. Also, the results of a paper by Chung concerns expander properties and the diameter [10], Green studied the clique number in [12] and Grynkiwicz et al. worked on connectivity of Cayley sum graphs [13]. To some extent, this situation may be explained by the fact that Cayley sum graphs are rather difficult to study. For example, it is uncomplicated to prove that any connected Cayley graph on a finite abelian group with at least three elements is hamiltonian; however, apart from the results of [9], nothing seems to be known on hamiltonicity of addition Cayley graphs on finite abelian groups. Also, the connectivity of a Cayley graph on a finite abelian group is easy to determine, while determining the connectivity of a Cayley sum graph is a non-trivial problem, see [13].

A graph is called integral if its adjacency matrix is an integral matrix, i.e. all its eigenvalues are rational integers. A finite group G is called $k-$integral, $1 \leq k \leq |G|$, if and only if for every subset S of G of size k, $\mathrm{Cay}^+(G, S)$ is integral. An interesting problem in algebraic graph theory is the characterizing integral graphs. The totally disconnected graph on n vertices, with spectrum $(0, \ldots, 0)$, and the complete graph on n vertices, with spectrum $(n-1, -1, \ldots, -1)$, are examples of integral. The notion of integral graphs dates back to Harary and Schwenk [14] who suggested the problem of determining which graphs have integral spectra? In general, the answer of this question appears to be difficult. Some authors investigated special classes of graphs including trees, graphs with bounded degrees, regular graphs, Cayley graphs, etc. For example, the path on n vertices is integral if and only if $n = 2$, also the cycle on n vertices is integral if and only if $n = 3, 4, 6$, see [7]. Bussemaker and Cvetković in [8] proved that among all connected cubic graphs, there are exactly 13 integral graphs. In [1], Abdollahi and Vatandoost determined all connected cubic integral

Cayley graphs. They proved that there are exactly seven connected cubic integral Cayley graphs.

A graph is a Cayley sum graph of a cyclic group if and only if it is an anti-circulant graph, a graph whose adjacency matrix is anti-circulant. The anti-circulant matrix associated to a vector v is an $n \times n$ matrix whose rows are given by iterations of the anti-shift operator acting on v.

A set $D \subseteq V$ of vertices in a graph $\mathcal{G} = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset is a dominating set. The domination number $\gamma(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum cardinality of a dominating set in $\mathcal{G}$ and the corresponding dominating set is called a γ-set.

A set $T \subseteq V(\mathcal{G})$ is a total dominating set if every vertex in $V(\mathcal{G})$ is adjacent to a vertex in T. The total domination number of a graph $\mathcal{G}$ denoted by $\gamma_t(\mathcal{G})$ is the minimum cardinality of all total dominating sets. Clearly, $\gamma(\mathcal{G}) \leq \gamma_t(\mathcal{G})$.

This thesis is organized as follows: In chapter 2, first we study some graph properties of Cayley sum graphs. Then by working on the concept of anti-circulant graphs, we obtain some results on the isomorphisms, connectivity and vertex transitivity on these graphs. Also, using the concept of Cayley sum color graphs and representation theory methods, we find the spectrum of Cayley sum color graphs of abelian groups and as a result we compute the spectrum of real anti-circulant matrices [4, 6]. Finally, we give some results on the (total) domination number of Cayley sum graphs [3].

In chapter 3, by studying on the integrality of Cayley sum graphs, first we show that the cyclic group of order 3 and all elementary abelian 2-groups are the only integral Cayley sum abelian groups. Also, we show that every elementary abelian 2-group and elementary abelian 3-group are the only 2-integral Cayley sum abelian groups, and every 3-integral Cayley sum abelian group is represented by $\mathbb{Z}_2^n \oplus \mathbb{Z}_3^m$ or $\mathbb{Z}_2^n \oplus \mathbb{Z}_4^m$, where $n \geq 0$ and $m \in \{0, 1\}$. As an important result, we prove that there are exactly six simple connected cubic integral Cayley sum graphs. In particular, if G is a finite arbitrary group, then for some square free subset S of size 3, $\mathrm{Cay}^+(G, S)$ is connected and integral if and only if G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^2,\ \mathbb{Z}_6,\ D_6,\ \mathbb{Z}_8,\ \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ \mathbb{Z}_2^3,\ \mathbb{Z}_{12},\ \mathbb{Z}_2 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_{12},\ \mathbb{Z}_4 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

For more details see [4] and [5].

Theoretical Background

First we recall a necessary and sufficient condition for a Cayley sum graph to be connected, proved by Lev [16].

LEMMA 1 [16] Let S be a subset of the finite abelian group G. In order for $\mathrm{Cay}^+(G, S)$ to be connected it is necessary and sufficient that S is not contained in a coset of a proper subgroup of G, except, perhaps, for the non-zero coset of a subgroup of index 2.

It follows from Lemma 1 that if $\mathrm{Cay}^+(G, S)$ is connected and S is contained in some non-zero coset of a proper subgroup H of G, then $|G : H| = 2$ and S is not contained in any other coset.

In [9], Cheyne et al. proved that if S is a square free subset of G, then $\mathrm{Cay}^+(G,S)$ is connected if and only if $G = \langle S \rangle$ and $|G : \langle S - S \rangle| \leq 2$. Also, they proved the following lemma.

LEMMA 2 [9] Let S be a square free subset of a finite abelian group G such that $\mathrm{Cay}^+(G,S)$ is connected. Then, the following statements are equivalent:

(i) $\mathrm{Cay}^+(G,S)$ is a bipartite graph;

(ii) $|G : \langle S - S \rangle| = 2$;

(iii) $S \cap \langle S - S \rangle = \emptyset$.

Furthermore, if $\mathrm{Cay}^+(G,S)$ is bipartite, then $\langle S-S \rangle$ and $G \setminus \langle S-S \rangle$ are the bipartition sets.

In this research, we use the concept of Cayley sum color graphs and representation theory methods to find the spectrum of Cayley sum color graphs.

For a finite group G, we denote by $\mathbb{C}[G]$ the complex vector space of dimension $|G|$ over the complex field $\mathbb{C}$ with a basis indexed by elements of G, namely $\{e_g\}_{g \in G}$. Thus, an element $\phi \in \mathbb{C}[G]$ has the form $\phi = \sum_{g \in G} \mu_g e_g$, where $\mu_g \in \mathbb{C}$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus, a function $\phi : G \to \mathbb{C}$ corresponds to the vector $\phi = \sum_{g \in G} \phi(g) e_g$ and vice versa. In particular, the vectors $\{e_g\}_{g \in G}$ of the standard basis correspond to the functions

$$e_g(h) = \begin{cases} 1 & \text{if} \quad g = h \\ 0 & \text{if} \quad g \neq h. \end{cases}$$

DEFINITION 3 Let d be a positive integer and $\mathrm{GL}_d(\mathbb{C})$ be the group of invertible matrices on $\mathbb{C}$. A representation of a group G of degree d is a homomorphism $\rho : G \to \mathrm{GL}_d(\mathbb{C})$.

For any $g \in G$, we denote by ρ_g the image of g in $\mathrm{GL}_d(\mathbb{C})$ and for any $v \in \mathbb{C}^d$, we denote by $\rho_g v$ the image of v under the action of ρ_g.

Recall that a complex square matrix M is unitary if $MM^* = M^*M = I$, where M^* denotes the conjugate transpose of M. In particular, if $T = [t_{ij}]$ is unitary, then the inverse of $[t_{ij}]$ is simply its complex conjugate transpose $[\overline{t}_{ji}]$, where $\overline{z}$ denotes the complex conjugate of a complex number z.

To find the spectrum of Cayley sum graphs, we use the following useful theorem, due to Issacs:

THEOREM 4 [15] Every linear representation $G \to \mathrm{GL}_d(\mathbb{C})$ of G is similar to a linear representation $\rho : G \to \mathrm{GL}_d(\mathbb{C})$ such that $\rho(g)$ is unitary, for all $g \in G$.

Results and Discussions

We commence with some elementary properties of $\mathrm{Cay}^+(G,S)$.

THEOREM 5 Let S be a normal subset of a non-trivial group G. Then, $\mathrm{Cay}^+(G,S)$ is connected if and only if one of the following statements holds:

(i) S is not contained in a coset of a proper subgroup of G;

(ii) S is contained in the non-trivial coset of an index 2 subgroup of G, but not contained in any other coset.

Using Theorem 5, we prove the same result of [9] in the general case as follows

THEOREM 6 Let S be a normal subset of a finite group G. Then, $\mathrm{Cay}^+(G,S)$ is connected if and only if $G = \langle S\rangle$ and $|G : \langle S^{-1}S\rangle| \leq 2$.

In the following theorem, the components of Cayley sum graphs are determined. We denote by $C(x)$ the connected component containing a vertex x.

THEOREM 7 Let S be a subset of G and put $H := \langle S - S\rangle$. If x is an element of G with $2x \notin H + s$, for some $s \in S$, then $C(x) = H + x \cup H + s - x$ is a bipartite subgraph of $\mathrm{Cay}^+(G,S)$; while if $2x \in H + s$, then $C(x) = H + x$. Moreover, for any $y \in G$ with $2y \notin H + s$, the components $C(x)$ and $C(y)$ are isomorphic. Also, $\mathrm{Cay}^+(G,S)$ has

$$\frac{1}{2}\left(|G/H| + |A|\right)$$

connected components, where $A = \{H + x \mid 2x \in H + s, \text{ for some } s \in S\}$.

In what follows, we prove some basic facts on the (total) domination number of Cayley sum graphs.

THEOREM 8 Let G be a finite abelian group of order n and S be a subset of G. Then,

$$\lceil \frac{n}{|S|+1} \rceil \leq \gamma(\mathrm{Cay}^+(G,S)) \leq n - |S| + 1.$$

REMARK 9 The bounds for $\gamma(\mathrm{Cay}^+(G,S))$ obtained in Theorem 8, can not be reduced in general. More specifically, the lower bound is attained for the graph $\mathcal{G}_1 = \mathrm{Cay}^+(\mathbb{Z}_4, \{1,3\})$ and the upper bound is attained for the graph $\mathcal{G}_2 = \mathrm{Cay}^+(\mathbb{Z}_5, \{0,1,2,4\})$.

THEOREM 10 Let G be a finite abelian group of order n and S be a subset of G, such that $\mathrm{Cay}^+(G,S)$ has no component of size 1. Then,

$$\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S| + 1.$$

In the following theorem, we find a bound for the total domination number of a connected Cayley sum graph.

THEOREM 11 Let G be a finite abelian group of order n and S be a subset of G of size less than $n-1$, such that $\mathrm{Cay}^+(G,S)$ is connected. If $S \cap 2G = \emptyset$, then $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S|$ and if $S \cap 2G \neq \emptyset$, then $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S| + 1$.

Now, we find some results on $\gamma(\mathrm{Cay}^+(\mathbb{Z}_n, S))$ and $\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n, S))$, where $S = \{1, 2, \ldots, k\}$ and $1 \leq k \leq (n-1)/2$.

THEOREM 12 Let $\mathcal{G} = \mathrm{Cay}^+(\mathbb{Z}_n, \{1, 2, \ldots, k\})$, where n, k are positive integers with $1 \leq k \leq (n-1)/2$. Then, $\lceil \frac{n}{k+1} \rceil \leq \gamma(\mathcal{G}) \leq \lceil \frac{n}{k} \rceil$.

REMARK 13 The bounds are given in Theorem 12 are sharp. Consider $\mathcal{G}_1 = \mathrm{Cay}^+(\mathbb{Z}_{15}, \{1,2,3,4\})$ and $\mathcal{G}_2 = \mathrm{Cay}^+(\mathbb{Z}_8, \{1,2,3\})$. The lower and upper bound is attained for $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

THEOREM 14 Let $n \in \mathbb{N}$ and $S = \{1, 2, \ldots, k\}$ be a subset of $\mathbb{Z}_n$, where $k \in \mathbb{Z}_n$. Then,

$$\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n, S)) = \lceil \frac{n}{k} \rceil.$$

THEOREM 15 Let $n \in \mathbb{N}$ and $S = \{1, 2, \ldots, k\}$ be a subset of $\mathbb{Z}_n$, where $k \in \mathbb{Z}_n$. Then, for every $i = 0, 1, \ldots, k-1$, subset $T_i = T + i$ is a total dominating set of $\mathrm{Cay}^+(\mathbb{Z}_n, S)$, where $T = \{0, k, 2k, \ldots, (\lceil \frac{n}{k} \rceil - 1)k\}$.

In what follows, we obtain some results on the isomorphisms, connectivity and vertex transitivity on anti-circulant graphs.

PROPOSITION 16 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two anti-circulant graphs on n vertices with anti-circulant sets $S(\mathcal{G}_1)$ and $S(\mathcal{G}_2)$, respectively, such that $S(\mathcal{G}_2) \equiv S(\mathcal{G}_1) + 2m \pmod{n}$ and $m \in \{0, 1, \ldots, n-1\}$. Then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.

REMARK 17 The converse of proposition 16 does not necessary hold. For example, let $n = 4$ and consider two anti-circulant sets $S(\mathcal{G}_1) = \{0, 3\}$ and $S(\mathcal{G}_2) = \{0, 1\}$. The relabelling $(0, 1, 2, 3) \longrightarrow (0, 3, 2, 1)$ implies that $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic but, there is no $m \in \{0, 1, 2, 3\}$ such that $S(\mathcal{G}_2) \equiv S(\mathcal{G}_1) + 2m \pmod{4}$.

THEOREM 18 Every anti-circulant graph $\mathcal{G}$ of n vertices with the anti-circulant set S is connected if and only if $\gcd(\{n\} \cup (S - S)) \leq 2$.

PROPOSITION 19 Let $\mathcal{G} = \mathrm{Cay}^+(\mathbb{Z}_n, S(\mathcal{G}))$ such that $S(\mathcal{G}) \cap 2\mathbb{Z}_n \neq \emptyset$. If $\mathcal{G}$ is vertex transitive, then $2\mathbb{Z}_n \subseteq S(\mathcal{G})$.

A partial converse of Proposition 19 is the Proposition 20 below:

PROPOSITION 20 Let $\mathcal{G} = \mathrm{Cay}^+(\mathbb{Z}_n, S(\mathcal{G}))$ such that $S(\mathcal{G})$ be a closed subset of $\mathbb{Z}_n$ and $2\mathbb{Z}_n \subseteq S(\mathcal{G})$. Then, $\mathrm{Cay}^+(\mathbb{Z}_n, S(\mathcal{G}))$ is vertex transitive.

Now, we want to give the eigenvalues of Cayley sum graphs. First we need the following facts.

Let G be a finite group and $\mathrm{Cay}^+(G, \alpha)$ be a Cayley sum color graph on G. For all $g, h \in G$, we may view the adjacency matrix $A = [a_{gh}]$ of $\mathrm{Cay}^+(G, \alpha)$ as a linear map $A : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by

$$Ae_h = \sum_{g \in G} a_{gh} e_g.$$

The following theorem shows that there is a connection between Cayley sum color graphs and the representation theory.

THEOREM 21 The adjacency matrix A of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$ satisfies

$$A = \sum_{g \in G} \alpha(g) \rho_g,$$

where ρ_g is a linear map of G on $\mathbb{C}[G]$ such that $\rho_g e_h = e_{gh^{-1}}$.

THEOREM 22 Let $\{\rho_1, \rho_2, \ldots, \rho_n\}$ be a complete set of irreducible inequivalent representations of G, d_k be the degree of ρ_k, and $\varrho^{(k)}$ be a unitary matrix representation of ρ_k. Also, let $\varrho_{ij}^{(k)}(g)$ be the ij-th entry of $\varrho^{(k)}(g)$, $1 \leq i, j \leq d_k$, and $\varrho_{ij}^{(k)} = \sum_{g\in G} \varrho_{ij}^{(k)}(g)e_g$. If A is an adjacency matrix of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$, then for every $k = 1, \ldots, n$ and $1 \leq i, j \leq d_k$, we have

$$A\overline{\varrho}_{ij}^{(k)} = \sum_{l=1}^{d_k} \left(\sum_{g\in G} \alpha(g)\overline{\varrho}_{lj}^{(k)}(g) \right) \varrho_{li}^{(k)}.$$

THEOREM 23 Let A be an adjacency matrix of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$. If α is a class function, then for every $1 \leq k \leq m$ and $1 \leq i, j \leq d_k$, we have

$$A\overline{\varrho}_{ij}^{(k)} = \left(\frac{1}{d_k} \sum_{g\in G} \alpha(g)\ \overline{\chi_k(g)} \right) \varrho_{ji}^{(k)}.$$

COROLLARY 24 If $\varrho_{ii}^{(k)}$ is a real-valued vector, then $\frac{1}{d_k}\sum_{g\in G} \alpha(g)\overline{\chi_k(g)}$ is an eigenvalue of $\mathrm{Cay}^+(G, \alpha)$ and if $\varrho_{ii}^{(k)}$ is not a real-valued vector, then $\pm|\frac{1}{d_k}\sum_{g\in G} \alpha(g)\overline{\chi_k(g)}|$ are two eigenvalues of $\mathrm{Cay}^+(G, \alpha)$, for $k = 1, 2, \ldots, m$.

If G is a finite abelian group of order n, then for every $k = 1, \ldots, m$, we have $d_k = 1$ and $\varrho^{(k)}(g) = \chi_k(g)$. In this case, since $\mathrm{span}\{\mathrm{Re}(\overline{\beta}_k\chi_k), \mathrm{Im}(\overline{\beta}_k\chi_k)\} = \mathrm{span}\{\chi_k, \overline{\chi_k}\}$, by Corollary 24 we can determine the spectrum of $\mathrm{Cay}^+(G, \alpha)$. Clearly, if α is a characteristic function of a subset $S \subseteq G$, then the result of [11] on the spectrum of $\mathrm{Cay}^+(G, S)$ holds. Note that, for every $\lambda \in \mathbb{C}$, the equation $z^2\lambda = |\lambda|$ have two complex roots $\exp(\frac{\theta}{2}i)$ and $\exp((\pi + \frac{\theta}{2})i) = -\exp(\frac{\theta}{2}i)$, where θ is the argument of $\frac{|\lambda|}{\lambda}$.

From Corollary 24 we conclude that more tools are need to obtain all eigenvalues of a Cayley sum graph. The next theorem help us to say more about the eigenvalues of a Cayley sum graph.

THEOREM 25 Let A be an adjacency matrix of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$. If α is a class function, then for every $k = 1, \ldots, m$ and $1 \leq i, j \leq d_k$, we have

$$A^2\overline{\varrho}_{ij}^{(k)} = \frac{1}{d_k^2} \left(\sum_{t\in G} \alpha(t)\overline{\chi}^{(k)}(t) \right) \left(\sum_{h\in G} \alpha(h)\chi^{(k)}(h) \right) \overline{\varrho}_{ij}^{(k)}.$$

So, every vector in the orthogonal basis $\{\overline{\varrho}_{ij}^{(k)}\}$ of $\mathbb{C}[G]$ is an eigenvector of A^2.

COROLLARY 26 Let A be an adjacency matrix of a Cayley sum graph $\mathrm{Cay}^+(G, S)$, where S is a conjugacy class of G. Then, for an arbitrary element s of S, the eigenvalue associated with the eigenvector $\overline{\varrho}_{ij}^{(k)}$ is

$$\left(\frac{|S|\ |\chi^{(k)}(s)|}{d_k} \right)^2,$$

for every $1 \leq k \leq m$ and $1 \leq i, j \leq d_k$.

Note that μ is an eigenvalue of A^2 if there exists an eigenvalue λ of A such that $\mu = \lambda^2$. Therefore, the multiplicity of the eigenvalue 0 in A^2 is equal to the multiplicity of the eigenvalue 0 in A.

Now, we find the spectrum of anti-circulant matrices.

Let G be a finite abelian group. Devose et al. determined the spectrum of $\mathrm{Cay}^+(G,S)$, where S is a subset of G [11]. In the following theorem, using Theorem 23, we generalize their result and determine the spectrum of $\mathrm{Cay}^+(G,\alpha)$. Recall that all irreducible representations of a finite abelian group have degree 1. Thus, in a finite abelian group irreducible representation and irreducible character are the same.

THEOREM 27 Let $\mathrm{Cay}^+(G,\alpha)$ be a Cayley sum color graph on a finite abelian group G and $\{\chi_1,\chi_2,\ldots,\chi_n\}$ be a complete set of irreducible inequivalent representations (characters) of G. If χ_k is real-valued, then $\sum_{g\in G}\alpha(g)\chi_k(g)$ is an eigenvalue of $\mathrm{Cay}^+(G,\alpha)$ and if χ_k is not real-valued, then $\pm|\sum_{g\in G}\alpha(g)\chi_k(g)|$ are two eigenvalues of $\mathrm{Cay}^+(G,\alpha)$.

If we take α to be the charatersitic function on a subset $S\subseteq G$ we obtain the following corollary.

COROLLARY 28 [11] Let $\mathrm{Cay}^+(G,S)$ be a Cayley sum graph of a finite abelian group G and $\{\chi_1,\chi_2,\ldots,\chi_n\}$ be a complete set of irreducible inequivalent representations (characters) of G. If χ_k is real-valued, then $\sum_{g\in S}\chi_k(g)$ is an eigenvalue of $\mathrm{Cay}^+(G,S)$ and if χ_k is not real-valued, then $\pm|\sum_{g\in S}\chi_k(g)|$ are two eigenvalues of $\mathrm{Cay}^+(G,S)$.

An anti-circulant set $S(\mathcal{G})$ can help us to determine the spectrum of an anti-circulant graph $\mathcal{G}$. Generally, in the light of Theorem 27, we can compute all eigenvalues of a real anti-circulant matrix.

THEOREM 29 Let M be an $n\times n$ real anti-circulant matrix with the first row $(m_0,m_1,\ldots,m_{n-1})$ and $w=\exp(2\pi i/n)$, where $n\in\mathbb{N}$. Put

$$s=[\frac{n-1}{2}] \quad \text{and} \quad \lambda_j=\sum_{k=0}^{n-1}m_kw^{jk}, \quad j=0,1,\ldots,s.$$

If n is even, then the eigenvalues of M are

$$\lambda_0,\ \lambda_{n/2},\ \pm|\lambda_j|, \quad j=1,2,\ldots,s,$$

where $\lambda_{n/2}=\sum_{k=0}^{n-1}(-1)^km_k$; and if n is odd, then the eigenvalues of M are

$$\lambda_0,\ \pm|\lambda_j|, \quad j=1,2,\ldots,s.$$

Now, we want to prove that there are exactly six simple connected cubic integral Cayley sum graphs. In order to charactrize cubic connected integral Caylet sum graphs we need the classification of simple connected cubic integral graphs given by Bussemaker and Cvetković [8].

LEMMA 30 [8] There are exactly 13 connected cubic integral graphs, say $\Gamma_i, 1\leq i\leq 13$.

First we consider Cayley sum graphs over abelian groups. Some basic facts on the integrality of Cayley sum graphs over abelian groups are as follows:

THEOREM 31 A finite abelian group G is Cayley sum integral if and only if G is an elementary abelian 2-group or $G\cong\mathbb{Z}_3$.

LEMMA 32 A finite abelian group G is 2−integral if and only if $G \cong \mathbb{Z}_2^n$ or $G \cong \mathbb{Z}_3^n$, for all $n \geq 0$.

LEMMA 33 If G is 3−integral, then $G \cong \mathbb{Z}_2^n \oplus \mathbb{Z}_3^m$ or $G \cong \mathbb{Z}_2^n \oplus \mathbb{Z}_4^m$, where $n \geq 0$ and $m \in \{0, 1\}$.

LEMMA 34 Let $\mathbb{Z}_n$ be a cyclic group of order $n, n \in \mathbb{N}$. Then there exists a subset S of $\mathbb{Z}_n$ of size two such that $\mathrm{Cay}^+(\mathbb{Z}_n, S)$ is connected and integral if and only if $n \in \{2, 3, 4, 6\}$.

LEMMA 35 Let $\mathbb{Z}_n$ be a cyclic group of order $n, n \in \mathbb{N}$. Then, there exists a square free subset S of $\mathbb{Z}_n$ of size three such that $\mathrm{Cay}^+(\mathbb{Z}_n, S)$ is connected and integral if and only if $n \in \{6, 8, 12\}$.

In the next theorem, we identify all simple connected cubic integral Cayley sum graphs over abelian groups.

THEOREM 36 There are exactly six simple connected cubic integral Cayley sum graphs over abelian groups. In particular, if G is a finite abelian group, then for some square free subset S of size 3, $\mathrm{Cay}^+(G, S)$ is connected and integral if and only if G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^2,\ \mathbb{Z}_6,\ \mathbb{Z}_8,\ \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ \mathbb{Z}_2^3,\ \mathbb{Z}_{12},\ \mathbb{Z}_2 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_{12},\ \mathbb{Z}_4 \oplus \mathbb{Z}_6,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

In the next step, we consider Cayley sum graphs over non-abelian groups and we show that there are one simple connected cubic integral Cayley sum graphs over non-abelian groups. First we need the following key lemmas.

LEMMA 37 Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1 \rangle$ be the dihedral group of order $2n$. Then, there exists a normal subset S of D_{2n} of size 3 such that $\mathrm{Cay}^+(D_{2n}, S)$ is a simple connected graph if and only if $n = 3$.

LEMMA 38 Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$, where $n \geq 3$. For every normal subset S of T_{4n} of size 3, $\mathrm{Cay}^+(T_{4n}, S)$ is not a connected graph.

LEMMA 39 Let $U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$, where $n \geq 3$. For every normal subset S of U_{6n} of size 3, $\mathrm{Cay}^+(U_{6n}, S)$ is not a connected graph.

LEMMA 40 Let S_n be the symmetric group on n letters. Then, there exists a normal subset S of S_n of size 3 such that $\mathrm{Cay}^+(S_n, S)$ is a simple connected graph if and only if $n = 3$.

Now we are ready to state the final result.

THEOREM 41 If G is a finite non-abelian group, then for some square-free normal subset S of size 3, $\mathrm{Cay}^+(G, S)$ is connected and integral if and only if G is isomorphic to D_6.

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